SOME PROBLEMS IN PRODUCTS' RELIABILITY MANAGEMENT

ABSTRACT
THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF

Doctor of Philosophy

IN

STATISTICS

By

MAROOF AHMAD KHAN

Under the Supervision of

DR. HASAN MATEEN-UL-ISLAM

DEPARTMENT OF STATISTICS & OPERATIONS RESEARCH
ALIGARH MUSLIM UNIVERSITY
ALIGARH (INDIA)

2006-2007

The research work presented in this thesis is spread over in six chapters. A comprehensive bibliography has also been given at the end, which has been referred during the research work.

Chapter I is expository in nature and provides a brief review of the concepts concerning reliability, availability, maintainability and the various associated aspects. Some basic definitions and lifetime distributions, which have been used, are discussed as well.

Chapter II deals with the problem of strength of a manufactured item with power function distribution, facing a Rayleigh stress. It is suggested that the item be so design that it has parameters with capability to meet the challenge with a given probability. And it has also been shown that it is better suited to face the stress that follows exponential distribution.

Chapter III addresses to the problem of strength of a manufactured item with Weibull distribution as stress distribution, and shows that it is most suitable to design a product facing power function as strength distribution, with the pre-determined probability and minimum cost too. Results obtained by Alam and Roohi (2003) and Khan and Islam (2007a) are the particular cases of the results obtained in this chapter.

Chapter IV focuses on the problem of strength of manufactured item against an array of stresses, treating it as a system. Reliability of the system is obtained, when $n-$Stresses acted on a single strength component with exponential probability distributions. Thus the stress components have been decomposed in the form of a multi-component system.

Chapter V discusses an alternative method, to evaluate reliability of system in stress-strength situation for $\alpha-$distributed strength and stress using, Gaussian function. Some of its variants are also discussed.

Chapter VI deals with the system amenable to maintenance and provides posterior analysis, Bayesian point estimators and Bayesian analysis of system availability with geometric failure as well as repair time distribution.

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Dr. Hasan M. Islam
(Reader)

March 15, 2007

CERTIFICATE

This is to certify that **Mr. Maroof A. Khan** has carried out the work reported in the present thesis entitled ***"Some Problems in Products' Reliability Management"*** under my supervision. The thesis is Mr. Khan's original work and I recommend it for consideration for the award of degree of "Doctor of Philosophy in Statistics". This work has not been submitted to any other university for the award of any other degree.

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Supervisor

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ACKNOWLEDGEMENTS

A Ph.D thesis is both an excruciating and enjoyable experience. This thesis is the product of many years of work, procrastination, changing minds and opinions and a colossal amount of external assistance. During these years, I have always dreamt that one day I would write the acknowledgement of my thesis, which would mean that it was nearly over. Now this moment has arrived and very crucial for me how to start. There have been many people supporting me during these days, in every different ways. I express my sincere gratitude and appreciation to all people who supported me energetically during the exciting and productive phases of this thesis. Without their kind help I could not have finished my work successfully. I extend my thanks to all of you.

In the first place, I wish to record my gratitude to ***Dr. Hasan Mateen-ul Islam*** *(Reader) for his supervision, advice, expert guidance and mentorship, his patience, encouragement and continuous support since the days I began work on. He has supported me academically and emotionally through the rough road to finish this thesis. He helped me come up with this thesis topic and guided me over the years. And during the most difficult times when writing this thesis, he gave me the moral support and the freedom I needed to move on. Above all and the most needed he provided me unflinching stimulations and support in various ways.*

Words fail me to express my appreciation to ***Prof. A.H. Khan****, one of the most eminent members of the faculty in the department of Statistics & Operations Research. During 'Master of Philosophy' level research, I had an opportunity to work with such a senior and renowned scholar. I am sure that there are only few students who ever have the pleasure of working for as a wonderful researcher and an educator as I have had. The light of the day comes in because early stage of this research with him gives me extra ordinary experiences. His truly scientist intuition has made him as a constant oasis of ideas and passions in science, which exceptionally inspire and enrich my growth as a student, want to be. I am indebted him more than he knows.*

It is pleasure for me to pay a tribute to **Dr. S.N. Alam**, the person who stands with me besides my advisor, in every decisive moments with charity and fervent, a distinguished former faculty member of the department of Statistics and Operations Research. My earnest confession must go humbly to him. He was always there to meet and talk about my ideas, to mark up my papers and chapters and raise questions to help me think through my problems when I doubted myself, brought good ideas in me. His many helpful comments were worth gold. He gave this thesis an artistic appearance. Hopefully, the groovy outside will stimulate people to also take glimpse at the inside. Thanks a lot, using your precious time to read this thesis and making it deserved special mention.

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Then I wish to thank **Prof. M.Z. Khan, Chairman,** Department of Statistics and Operations Research to provide for, fascinating services to the research scholars. Also Prof. M.J. Ahsan, Dr. M. Yaqub, Dr. R.U. Khan and other faculty members of the department who's numerous discussions about my works. Their admirable calmness even in stressful phases and often curious humour helped me to overcome every difficulty.

I feel truly honoured to squeeze out my thankfulness towards the non-teaching staff and seminar staff as well, for their words of interest, support and collaboration through out this time with me.

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Without any doubt, the person to whom I am most grateful is my father, Dr. Shakeel Ahmad. He always stood by my side. His drives unconditional support and belief in me have been really decisive. To my father I owe everything. He always encouraged me to fulfill my dreams and stoically bore his separation and gave me all the support in the world. No body is going to live this moment of seeing my thesis finished more happily than him. It is for you!

Finally I would like to thank everybody who is important to the successful realization of thesis, as well as expressing my apology that I could not mention personally one by one.

I dedicate this thesis to my late mother who has always been around me with her blessings.

March 15, 2007

(Maroof A. Khan.)

CONTENTS

Preface

Nearly everyday we learn of another company that has failed. The mind-boggling rate of industrial expansion of the past few decades has produced innumerable technical devices and systems on which we rely on our daily life for modern convenience, safety, and some times even preservation of human lives. In the new millennium, this rate of failure will increase. Competitors are rapidly entering the market place using latest technology, innovation and reliability on their products to gain market share. Profit margins are shrinking. Internet shopping challenges the conventional business model. The information highway is changing the way consumers make buying decision. Consumers have more resources availability for product information, bringing them new awareness about product reliability.

Today's engineering systems have become increasingly complex to design and build while the demand for reliability, quality and cost effective development continues. Reliability is one of the most important attributes in such critical systems as defense systems, aerospace applications, real time controls, medical applications as well as commercial systems. Growing international competition has increased the need for all engineers and designers to ensure an optimum level of quality and reliability of their products at the lowest cost. Hence the interest in reliability and quality has been growing in recent years.

These changes have made it easier for consumers to choose the best product for their individual needs. As better-informed shoppers, consumers can now determine their product needs at any place, any time, and for the best price. The information age allows today's consumer to research an entire market efficiency at any time and with little effort.

Conventional shopping is being replaced by 'smart' shopping. And a big part of smart shopping is getting the best product for the best price.

As the resources for product information continue to increase, the information available about the quality of the product increases as well. In the past, information on product quality was available through consumer magazines, newspapers, and television. The information was not always current and often did not cover the full breadth of the market. Today's consumer is using global information sources and Internet chat to help in their product selection process. An important part of the consumer's selection process is information regarding product's quality and reliability. Does it really do what the manufacturer claims? Is it easy to use? Is it safe? Will it meet consumer expectations of trouble free use? The list can be very long and very specific to the individual consumer.

In today's marketplace, product quality is necessary in order to stay in business. In tomorrow's marketplace, reliability will be the norm. Quality and reliability are terms that are often used interchangeably. While strongly connected, they are not the same. In the simplest terms:

- Quality is conformance to specifications.
- Reliability is conformance to specification over time.

Reliability is the continuation of quality over time. It is simply the time period over which a product meets the standards of quality for the period of expected use. Quality is now the standard for doing business. In today's marketplace and beyond, reliability will be the standard for doing business. The quality revolution is not over; it has just evolved into the reliability revolution, giving rise to so many questions. The present thesis is an attempt to answer some of the relevant questions

This thesis entitled **"Some problems in products' reliability management"** consists of six chapters, in which **Chapter I** is introductory in nature and deals with the basic concepts of reliability and the various associated aspects.

Chapter II deals with the problem of strength of a manufactured item following power function distribution and it has been shown that it is better suited to face the stress that follows Rayleigh distribution as compared to the stress that follows exponential distribution.

Chapter III addresses to the problem of strength of a manufactured item with Weibull distribution as stress distribution, and shows that it is most suitable to design a product facing power function as strength distribution, with the pre-determined probability and minimum cost too.

Chapter IV focuses on the problem of strength of manufactured item against an array of stresses, treating it as a system. Reliability of the system is obtained, when $n-$Stresses acted on a single strength component with exponential probability distributions. Thus the stress components have been decomposed in the form of a multi-component system.

Chapter V discusses an alternative method, to evaluate reliability of a system in stress-strength situation for $\alpha-$distributed strength and stress, using Gaussian function. Some of its variants are also discussed.

Chapter VI deals with the system amenable to maintenance and provides Bayesian analysis of system availability with geometric failure as well as repair time distribution.

We have appended a comprehensive bibliography of literature which have been referred to and/or are related to the material presented in the thesis.

LIST OF APPENDED PAPERS

Khan, M.A. and Islam, H.M. (2007a): On facing Rayleigh stress with strength having power function distribution. *Journal of Applied Statistical Science*, USA. Vol. **16** (*To appear*).

Khan, M.A. and Islam, H.M. (2007b): On stress and strength having power function distribution. *Pakistan Journal of Statistics*, Pakistan. Vol. **23**, 83-88.

Islam, H.M. and Khan, M.A. (2007a): On system reliability with single strength and multi-component stress model. *Submitted for publication.*

Khan, M.A. and Islam, H.M. (2007c): A note on reliability computation for $\alpha-$distributed strength and stress using Gaussian function. *Submitted for publication.*

Islam, H.M. and Khan, M.A. (2007b): Bayesian analysis of system availability with geometric failure law in life testing. *Submitted for publication.*

BASIC CONCEPTS OF RELIABILITY MANAGEMENT

1. Introduction

Since the beginning of civilization, humanity has attempted to predict the future. Watching the flight of birds, the movement of the leaves on the trees and other methods were some of the practices used. Fortunately, today's engineers do not have to depend on a crystal ball in order to predict the "future" of their products. Through the product life-data analysis, reliability engineers determine the probability and capability of parts, components, and systems to perform their required functions for desired periods of time without failure, in specified environments.

Life-data can be lifetimes of products in the marketplace, such as the time the product operated successfully or the time the product operated before it failed. These lifetimes can be measured in hours, miles, cycles-to-failure, stress cycles or any other metric with which the life or exposure of a product can be measured. All such data of product lifetimes can be encompassed in the term life-data or, more specifically, product life-data. The subsequent analysis and prediction are described as life data analysis. For this purpose, we will limit our examples and discussions to lifetimes of inanimate objects, such as equipment, components and systems, as they apply to reliability engineering.

Before performing life data analysis, the failure mode and the life units (hours, cycles, miles, etc.) must be specified and clearly defined. Further, it is quite necessary to define exactly what constitutes a failure. In other words, before performing the analysis it must be clear when the product is considered to have actually failed. This may seem rather obvious, but it is not uncommon for problems with failure definitions or time unit discrepancies to completely invalidate the results of expensive and time-consuming life testing and analysis.

2. What is reliability?

Reliability is a broad term that focuses on the ability of a product to perform its intended function. Mathematically speaking, assuming that an item is performing its intended function at time zero, reliability can be defined as the probability that it will continue to perform its intended function without failure for a specified period of time under stated conditions. Please note that the product defined here could be an electronic or mechanical hardware product, a software product, a manufacturing process or even a service.

In other words, reliability is the probability of success at a specified age under specified conditions. Success is defined by customers. Age or operating time is measured in hours, days, and years or in terms of cycles or some other appropriate measure. Relevant conditions are field conditions, not laboratory conditions or demonstration test conditions. Field reliability is what happens actually in real life.

3. Importance of reliability

There are a number of reasons why reliability is an important product attribute, including:

- **Reputation:** A company's reputation is very closely related to the reliability of its products. The more reliable a product is, the more likely the company is to have a favourable reputation.
- **Customer satisfaction:** While a reliable product may not dramatically affect customer satisfaction in a positive manner, an unreliable product

will negatively affect customer satisfaction severely. Thus high reliability is a mandatory requirement for customer satisfaction.

- **Warranty costs:** If a product fails to perform its function within the warranty period, not only the replacement and repair costs will negatively affect profits, there may be an unwanted negative publicity. Introducing reliability analysis is an important step in taking corrective action, ultimately leading to a product that is more reliable.
- **Repeat business:** A concerted effort towards improved reliability shows existing customers that a manufacturer is serious about its product and committed to customer satisfaction. This type of attitude has a positive impact on future business.
- **Cost analysis:** Manufacturers may take reliability data and combine it with other cost information to illustrate the cost-effectiveness of their products. This life cycle cost analysis can prove that although the initial cost of a product might be higher, the overall lifetime cost is lower than that of a competitor's because their product requires fewer repairs or less maintenance.
- **Customer requirements:** Many customers in today's market demand that their suppliers have an effective reliability program. These customers are conscious of the benefits of reliability analysis from their own experiences.
- **Competitive advantage:** Many companies will publish their predicted reliability numbers to help gain an advantage over their competitors who either do not publish their numbers or have lower numbers.

4. Reliability engineering

Reliability engineering consists of the systematic application of time-honored engineering principles and techniques throughout a product lifecycle and is thus an essential component of a good Product Lifecycle

Management program. The goal of reliability engineering is to evaluate the inherent reliability of a product or process and pinpoint potential areas for reliability improvement. Realistically, all failures cannot be eliminated from a design, so another goal of reliability engineering is to identify the most likely failures and then identify appropriate actions to mitigate the effects of those failures.

The reliability evaluation of a product or process can include a number of different reliability analyses. Depending on the phase of the product lifecycle, certain types of analysis are appropriate. As the reliability analysis is being performed, it is possible to anticipate the reliability effects of design changes and corrections. The different reliability analyses are all related, and examine the reliability of the product or system from different perspectives, in order to determine possible problems and assist in analyzing corrections and improvements.

Reliability engineering can be done by a variety of engineers, including reliability engineers, quality engineers, test engineers, systems engineers or design engineers. In highly evolved teams, all key engineers are aware of their responsibilities in regards to reliability and work together to help improve the product.

The reliability engineering activity should be an ongoing process starting at the conceptual phase of a product design and continuing throughout all phases of a product lifecycle. The goal always needs to be to identify potential reliability problems as early as possible in the product lifecycle. While it may never be too late to improve the reliability of a product, changes to a design are orders of magnitude less expensive in the early part of a design phase rather than once the product is manufactured and in service.

5. Reasons for reliability engineering

- For a company to succeed in today's highly competitive and technologically complex environment, it is essential that it knows the reliability of its product and is able to control it in order to produce products at an optimum reliability level. This yields the minimum life-cycle cost for the user and minimizes the manufacturer's costs of such a product without compromising the product's reliability and quality.
- Our growing dependence on technology requires that the products that make up our daily lives successfully work for the desired or designed-in period of time. It is not sufficient that a product works for time shorter than its mission duration, but at the same time there is no need to design a product to operate much past its intended life, since this would impose additional costs on the manufacturer. In today's complex world where many important operations are performed with automated equipment, we are dependent on the successful operation of these equipment (i.e. their reliability) and, if they fail, on their quick restoration to function (i.e. their maintainability).
- Product failures have varying effects, ranging from those that cause minor nuisances, such as the failure of a television's remote control (which can become a major nuisance, if not a catastrophe, depending on the football schedule of the day), to catastrophic failures involving loss of life and property, such as an aircraft accident. Reliability Engineering was born out of the necessity to avoid such catastrophic events and, with them, the unnecessary loss of life and property. It is not surprising that Boeing was one of the first commercial companies to embrace and implement reliability engineering, the success of which can be seen in the safety of today's commercial air travel.

- Today, reliability engineering can and should be applied to many products. The previous example of the failed remote control does not have any major life and death consequences to the consumer. However, it may pose a life and death risk to a non-biological entity: the company that produced it. Today's consumer is more intelligent and product-aware than the consumer of years past. The modern consumer will no longer tolerate products that do not perform in a reliable fashion, or as promised or advertised. Customer dissatisfaction with a product's reliability can have disastrous financial consequences to the manufacturer. Statistics show that when a customer is satisfied with a product he might recommend it to a few of his acquaintances; however, a dissatisfied customer will go public and criticize it vociferously.
- The critical applications with which many modern products are entrusted make their reliability a factor of paramount importance. For example, the failure of a computer component will have more negative consequences today than it did twenty years ago. This is because twenty years ago the technology was relatively new and not very widespread, and one most likely had backup paper copies somewhere. Now, as computers are often the sole medium in which many clerical and computational functions are performed, the failure of a computer component will have a much greater effect.

6. A few common sense applications

The reliability 'Bathtub' curve

Most products (as well as humans) exhibit failure characteristics as shown in the bathtub curve of Figure 1.1.

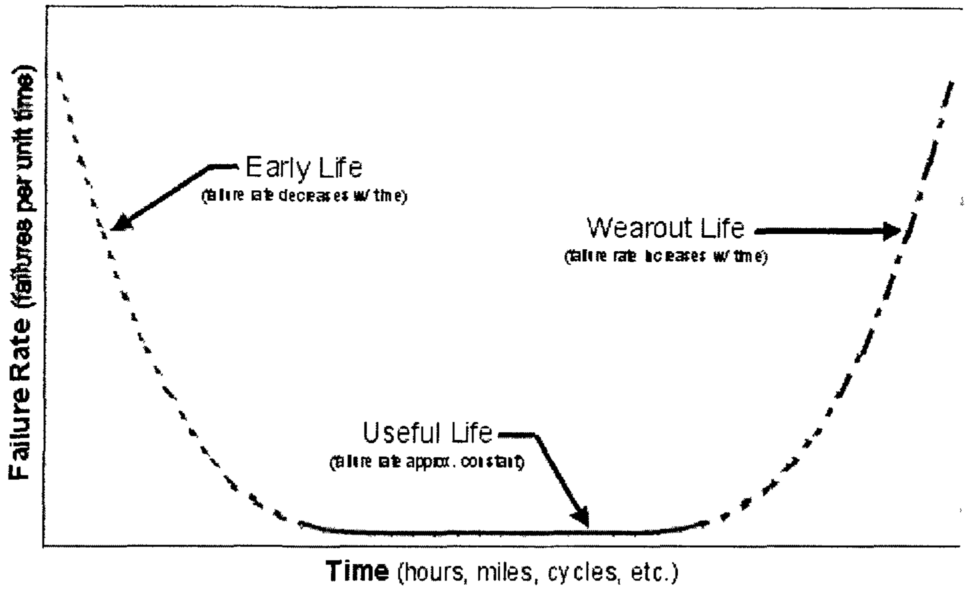

Figure 1.1: An idealized reliability bathtub curve, with the three major life regions: early, useful and wearout.

This curve is plotted with the product life on the x-axis and with the failure rate on the y-axis. The life can be in minutes, hours, years, cycles, actuations or any other quantifiable unit of time or use. The failure rate is given as failures among surviving units per time unit. As can be seen from this figure, many products will begin their lives with a higher failure rate (which can be due to manufacturing defects, poor workmanship, poor quality control of incoming parts, etc.) and exhibit a decreasing failure rate. The failure rate then usually stabilizes to an approximately constant rate in the useful life region, where the failures observed are chance failures. As the products experience more use and wear, the failure rate begins to rise as the population begins to experience failures related to wear-out. In the case of human mortality, the mortality rate (failure rate) is higher during the first year or so of life, then drops to a low constant level during our teens and early adult life and then rises as we progress in years.

Burn-In

Looking at this particular bathtub curve, it should be fairly obvious that it would be best to ship a product at the beginning of the useful life region, rather than right off the production line; thus preventing the customer from experiencing early failures. This practice is what is commonly referred to as burn-in, and is frequently performed for electronic components. The determination of the correct burn-in time requires the use of reliability methodologies, as well as optimization of costs involved (*i.e.* costs of early failures vs. the cost of burn-in), to determine the optimum failure rate at shipment.

Minimizing the manufacturer's cost

Figure 1.2 shows the product reliability on the x-axis and the producer's cost on the y-axis.

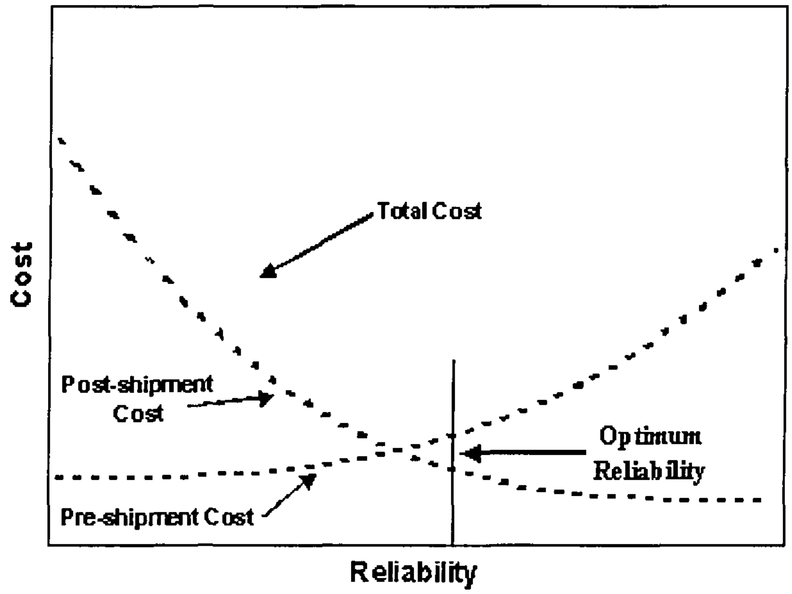

Figure 1.2: Total product cost vs. product reliability.

If the producer increases the reliability of his product, he will increase the cost of the design and/or production of the product. However, a low production and design cost does not imply a low overall product cost. The overall product cost should not be calculated as merely the cost of the product when it leaves the shipping dock, but as the total cost of the product through its lifetime. This includes warranty and replacement costs for defective products, costs incurred by loss of customers due to defective products, loss of subsequent sales, etc. By increasing product reliability, one may increase the initial product costs, but it will most likely decrease the support costs. An optimum minimal total product cost can be determined and implemented by calculating the optimum reliability for such a product. Figure 1.2 depicts such a scenario. The total product cost is the sum of the production and design costs as well as the other post-shipment costs. It can be seen that at an optimum reliability level, the total product cost is at a minimum. The optimum reliability level is the one that coincides with the minimum total cost over the entire lifetime of the product.

7. Disciplines covered by reliability engineering

Reliability engineering covers all aspects of a product's life, from its conception, subsequent design and production processes, through its practical use lifetime, with maintenance support and availability. Reliability engineering covers:

- Reliability.
- Maintainability and
- Availability.

All three of these areas can be numerically quantified with the use of reliability engineering principles and life-data analysis. Reliability having been defined earlier, we now discusses the maintainability and availability.

Maintainability

When a piece of equipment has failed, it is important to get it back into an operating condition as soon as possible, this is known as maintainability. For a given active maintenance action, the maintainability of a system is defined as the probability that it can be retained in or restored to a specific condition at a given time.

Maintainability is the probability that a given active maintenance action for an item under given conditions of use can be carried out within a stated time interval, when the maintenance is performed under stated conditions and using stated procedures and resources. The purpose of maintainability engineering is to increase the efficiency and safety and to reduce the cost of equipment maintenance, when maintenance is performed under given conditions and using stated procedures and resources.

Maintainability requirements must be:

i). Initially planned for and included within the overall planning documentation for a given program or project;

ii). Specified in the top level specification for the applicable system/product;

iii). Designed through the iterative process of functional analysis, requirements allocation, trade-off and optimization, synthesis and component selection and

iv). Measured in terms of adequacy through system test and evaluation.

Maintainability, defined in the broadest sense, can be measured in term of a combination of different maintenance factors. From a system perspective, it is assumed that maintenance factors can be broken down into the following general categories:

a). Corrective maintenance: Unscheduled maintenance accomplished, as a result of failure, to restore a system or product to a specified level of performance. This includes the possible ongoing modification of software to bring it to the proper operational state in the event that it has not achieved the desired level of maturity when the system is delivered to the customer.

b). Preventive maintenance: Scheduled maintenance accomplished to retain a system at a specified level of performance by providing systematic inspection, detection, servicing or prevention of impending failures through periodic item replacement.

Maintenance downtime constitutes the total elapsed time required (when the system is not operational) to repair and restore a system to full operating status, or retain a system in that condition. Figure below illustrates the relationship of the various downtime factors within the context of the overall time domain.

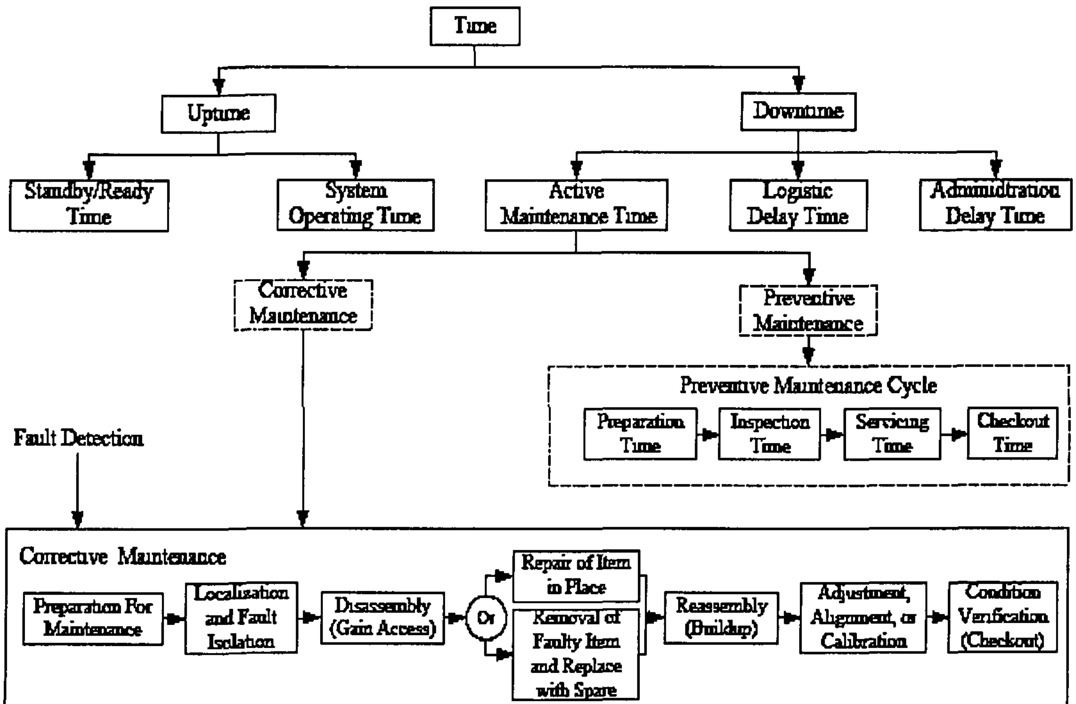

Figure 1.3: Composite view of uptime/downtime factors (Blanchard and Fabrycky, 1998)

In order to increase maintainability, in some manner the repair time must be reduced. There are several key concepts that should be followed as part of any design actively that support this reduction. The inner circle in figure 1.4 identifies inherent maintainability design features, and outer circle list secondary factors affecting maintainability focus on the maintenance and supply resources necessary to support the repair process. Establishing and maintaining the proper levels of these resources is often considered part of the logistic process.

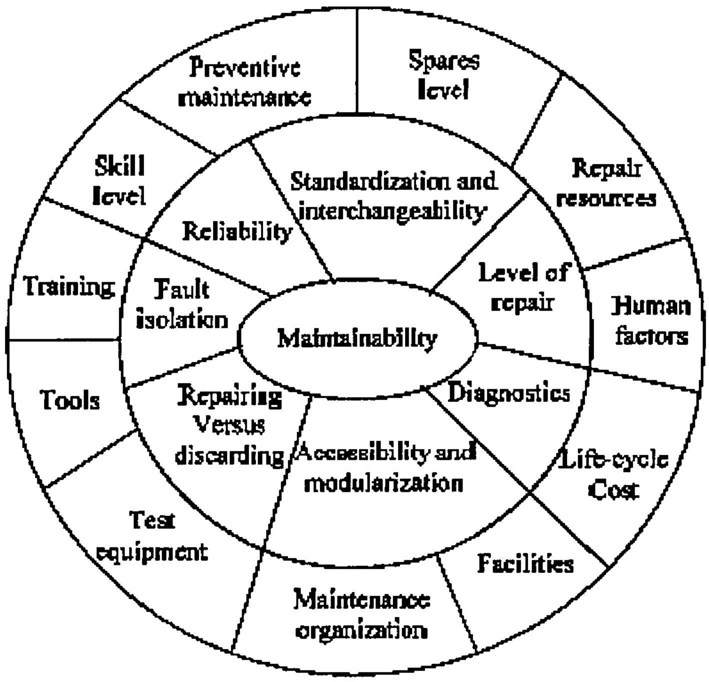

Figure 1.4: Inherent and secondary maintainability design features (Ebeling, 1997)

Availability

Availability is the probability that a system is performing its required function at a given point in time or over a stated period of time when operated and maintained in a prescribed manner (Ebeling, 1997). Like reliability, availability is a probability. Consider a system (device) which can be in one of two states, namely 'up (on)' and 'down (off)'. By 'up' we mean that the system is still functioning and by 'down' we mean that the system is not functioning, in the latter case it is being repaired or replaced, depending on whether the system is repaired or not. Let the state of the system be given by a binary variable

$$X(t)=\begin{cases}1, & \text{if the system is up at time t}\\ 0, & \text{otherwise}\end{cases}$$

An important characteristic of a repairable system is availability. Barlow and Proschan (1975) define four measures of availability performance: the availability function, limiting availability, the average availability function and limiting average availability. All of these measures are based on the function $X(t)$, which denotes the status of a repairable system at time t. The instant availability at time t (or point availability) is defined by: $A(t)=P(X(t)=1)$. This is the probability that the system is operational at time t. Because it is very difficult to obtain an explicit expression for $A(t)$, other measures of availability have been proposed. One of these measures is the steady system availability (or steady state availability, or limiting availability) of a system which is defined as

$$A=\lim_{t\to\infty} A(t)$$

This quantity is the probability that the system will be available after it has been run for a long time, and is a very significant measure of performance of a repairable system. There are several different forms of the steady state availability depending upon the definition of uptime and downtime. Some of these definitions are discussed in the following:

a). Inherent availability: inherent availability is the probability that a system or equipment, when used under stated conditions, in an ideal support environment (i.e., readily available tools, spares, maintenance personnel, etc) which will operate satisfactorily at any point in time as required (Blanchard and Fabrycky, 1998). It excludes preventive or scheduled maintenance action, logistic delay time and administrative delay time and is expressed as

$$A_i = \lim_{t \to \infty} A(t) = \frac{MTBF}{MTBF + MTTR}$$

Inherent availability is based solely on the failure distribution and repair time distribution. it can be therefore viewed as an equipment design parameter, and reliability-maintainability trade-off can be based on this interpretation (Ebeling, 1997).

b). Achieved availability: achieved availability is the probability that a system or equipment when used under stated conditions in an ideal support environment (i.e., readily available tools, spares, personnel, etc) which will operate satisfactorily at any point in time (Blanchard and Fabrycky, 1998). The achieved availability is defined as

$$A_a = \frac{MTBM}{MTBM + \overline{M}}$$

where the mean time between maintenance (MTBM) operations includes both unscheduled and preventive maintenance and $\overline{M}$ is the mean active maintenance time. If it is performed too frequently, preventive maintenance can have a negative impact on the achieved availability even though it may increase the MTBF.

c). Operational availability: operational availability is the probability that a system or equipment, when used under stated conditions in an actual operational environment, will operate satisfactorily when called upon (Blanchard and Fabrycky, 1998). The operational availability is defined as

$$A_o = \frac{MTBM}{MTBF + MDT}$$

where MDT is the mean maintenance downtime and includes maintenance time $(\overline{M})$, logistics delay time and administrative delay time.

8. Reliability and quality

Even though a product has a reliable design, when the product is manufactured and used in the field, its reliability may be unsatisfactory. The reason for this low reliability may be that the product was poorly manufactured. So, even though the product has a reliable design, it is effectively unreliable when fielded, which is actually the result of a substandard manufacturing process. As an example, cold solder joints could pass initial testing at the manufacturer level, but may fail in the field as the result of thermal cycling or vibration. This type of failure does not occur because of an improper design, but rather it is the result of an inferior manufacturing process. So while the product may have a reliable design, its quality is unacceptable because of the inferior manufacturing process.

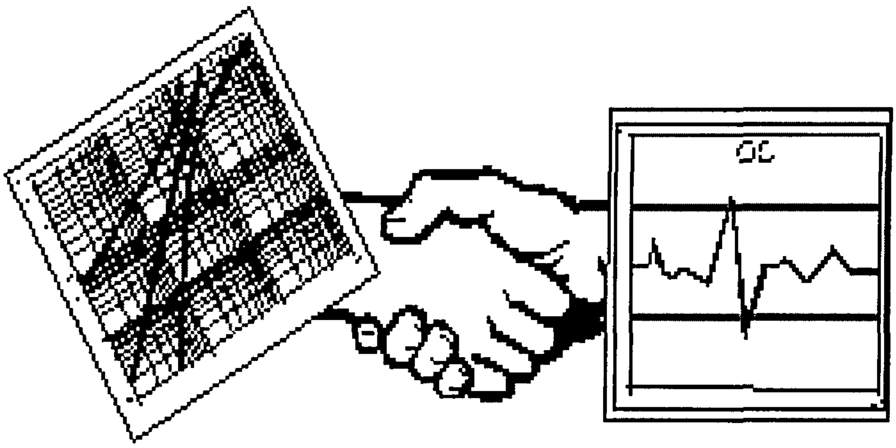

Figure 1.5: Reliability meets quality

Although the terms reliability and quality are often used interchangeably, there is a difference between these two concepts. While reliability is concerned with the performance of a product over its entire lifetime, quality control is concerned with the performance of a product at one point in time, usually during the manufacturing process. As stated in the definition, reliability assures that components, equipment and systems function without failure for desired periods during their whole design life, from conception (birth) to junking (death) stage. Quality control is a single, albeit vital, link in the total reliability process. Quality control assures conformance to specifications. This reduces manufacturing variance, which can degrade reliability. Quality control also checks that the incoming parts and components meet specifications, that products are inspected and tested correctly, and that the shipped products have a quality level equal to or greater than that specified. The specified quality level should be one that is acceptable to the user, the consumer and the public. No product can perform reliably without the inputs of quality control because quality parts and components are needed to go into the product so that its reliability is assured.

Just like a chain is only as strong as its weakest link, a highly reliable product is only as good as the inherent reliability of the product and the quality of the manufacturing process.

9. Reliability and safety factor

As the complexity of equipment arrangements increases, the assessment of risk becomes more complicated. Risk should be measured relative to the ability of the plan to reliably meet its specific operating mission. Consequently, the expected return on investment is seen as being directly related to plant equipment capability, defined in terms of, durability, performance, availability and reliability. It is clear that availability and reliability continues to be significant issue for industry. In industries like electronics, aviation and space, weight and performance are major factors. The need for additional fuel for space mission is many times greater than the weight of the traditional payload as each gram of payload costs dearly. Generally, in such systems, the safety factor is as little as 10%. If such low factors are to be used, knowledge of the distribution of stresses and strength and their relationship is essential. If the probability distribution for strength and stress are exactly known or may be approximated by some well-known distribution such as exponential, Weibull and normal etc, then the safety factor can easily be defined mathematically.

10. Reliability engineering and business plans

Reliability engineering assessment is based on the results of testing from in-house (or contracted) labs and data pertaining to the performance results of the product in the field. The data produced by these sources are utilized to accurately measure and improve the reliability of the products being produced. This is particularly important as market concerns drive a constant push for cost reduction. However, one must be able to keep a perspective on "the big picture" instead of merely looking for the quick fix. It is often the temptation to cut corners and save initial costs by using cheaper parts or cutting testing programs. Unfortunately, cheaper parts are usually less reliable and inadequate testing programs can allow products with undiscovered flaws to get out into the field. A quick savings in the short term by the use of cheaper components or small test sample sizes will usually result in higher long-term costs in the form of warranty costs or loss of customer confidence. The proper balance must be struck between reliability, customer satisfaction, time to market, sales and features. Figure 1.6 illustrates this concept. The polygon on the left represents a properly balanced project. The polygon on the right represents a project in which reliability and customer satisfaction have been sacrificed for the sake of sales and time to market.

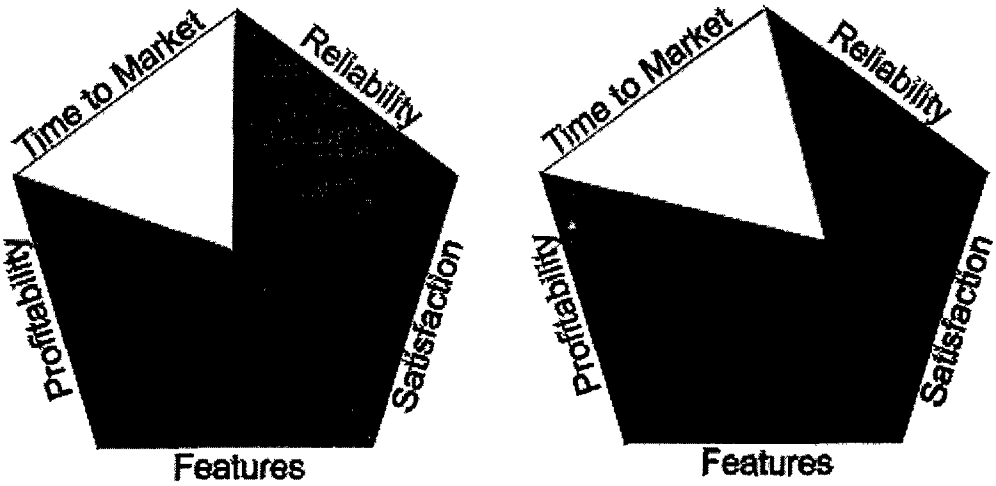

Figure 1.6: Graphical representation of balanced and unbalanced projects.

Through proper testing and analysis in the in-house testing labs, as well as collection of adequate and meaningful data on a product's performance in the field, the reliability of any product can be measured, tracked and improved, leading to a balanced organization with a financially healthy outlook for the future.

11. Stress-Strength models

A variable Y is said to be stochastically larger than a variable X if the cumulative distribution function of Y is never greater than that of X, i.e. using the standard notation

$$F_Y(t) \leq F_X(t) \text{ for all } t$$

an immediate consequence of this relation is that

$$P(X < Y) > \frac{1}{2}$$

However, if $F_X \equiv F_Y$, we have

$$P(X<Y)=\int_{-\infty}^{\infty} F_X(t)\,dF_Y(t) \geq \int_{-\infty}^{\infty} F_Y(t)\,dF_Y(t) = \frac{1}{2}$$

or, equivalently, $P(Y<X)=1/2$. Both these imply that out of two equivalent forces, probability that one exceeds the other is equal to ½. The problem of interest remains as to what happens when the two are not equal. If the two variables are designated as strength of an item Y and the stress it is likely to face (X), the problem of studying $P(X<Y)$ or $P(Y<X)$ gives rise to the 'Stress-Strength' models.

Motivations

In an important methodological note written about 30 years ago, Wolfe and Hogg (1971) assert that the numerical values of $P(X<Y)$ make more sense to practitioners- particularly those in medical profession – than the equivalent statement about $(\mu_1-\mu_2)/\sigma$ (under the normal assumption) and point out that $P(X<Y)$ can be estimated under many distributional assumptions (not only the normality) thus permitting us to avoid the trap of using normal distribution when they are obviously inappropriate. In a sense Wolfe and Hogg (1971) provide a road map to the research which resulted in a flood of papers starting from Church and Harris (1970) up to the beginning of the 21st century. Not only the problems of deriving theoretical expression for $P(X<Y)$ and its modifications and extensions under various distribution assumptions were found to be challenging, but also estimates of these probabilities based on the samples of various structures opened new avenues deriving approximations to variances and confidence bounds.

Similar sentiments were expressed some fifteen years later by Halperin *et al.* (1987) who emphasize the suitability of $P(X<Y)$ estimators for

versatile comparisons of two samples embracing the possibility that two underlying distributions may differ in one or more parameters.

It might be desirable to elaborate a bit on the assumption and characteristic of the pivotal quantities involved in the seemingly straightforward model: $P(X < Y)$.

In the seventies of the 20^{th} century when first serious attempts to analyze reliability of a component by applying probabilistic argument to a physical model of failure were initiated, the term "inference theory" was often used in engineering literature (Mazumdar, 1970). According to this theory if a component fails at any moment, the applied stress (often being a load) exceeds the components strength (or resistance). The stress –as we have already alluded –is a function of the environment in which the component is located and can be estimated from the available technological knowledge about the relevant conditions of the system and the manner in which they interact. Engineers claim (or used to claim) that the values of the mechanical stress at different points of time can be computed deterministically given the set of initial values. Church and Harris (1970) provide an example of the missile flight where the initial value of the stress corresponds to the propulsive force, angles of elevation, atmospheric condition, etc. One is tempted to recall the famous assertion of Laplace (1812) in his "Theory Analytique des Probabilities" to the effect that given the initial conditions and some relevant data one can predict, with complete certainty, the location of the moving particle at any given time. Laplace believes that "the curve describing a simple molecule of air or any gas is regulated in a manner as certain as the planetary orbits, the only difference between them lies in our ignorance". "Give me the sufficient data", -claimed Laplace- and "I will tell you the exact location of the ball on a Billiard table". A less rigid approach is to use random variables since, after all, even the initial conditions are random quantities. This amount to postulating stress to be a random variable is based on a priori conditions. The strength cannot be computed from a priori considerations and can be estimated by means of statistical methods from the results of the test specially designed for this purpose. This sets limitations on the amount of the data that can be generated and increases the temptation to use expert elicitation and Bayesian methods.

History

It may be of interest to point out that chronologically the stress-strength model originated not in a parametric but rather in a non-parametric set-up in the path breaking works of Wilcoxon (1945), Mann and Whitney (1947). The main objective of these investigations was to compare two random variables X and Y that describe results of two treatments. Wilcoxon, Mann and Whitney introduced statistics which bears their name and is based on ranks of the observations on X and Y in the joint sample. They also pointed out the connection between the hypothesis $F(X)=F(Y)$ and $P(X<Y)=1/2$. Their initial efforts led to the series of papers studying point and interval estimation of $P(X<Y)$ in the sixties of the last century. Other notable contributions in this area are these of Birnbaum (1956), Birnbaum and McCarty (1958), Govindarajulu (1967, 1968), Owen *et al.* (1964), Sen (1960, 1967), and Van Dantzig (1951) among others. Non-parametric methods were 'safe' in the sense that they imposed no assumptions on X and Y, however; they may be too inefficient for practical purposes.

The first attempt to study $P(X<Y)$ under certain parametric assumptions on X and Y was undertaken by Owen *et al.* (1964) that constructed confidence limits for $P(X<Y)$ when X and Y are

dependent or independent normally distributed random variables. In the sixties very little was done to investigate a parametric version of the stress-strength model, however in the seventies investigation of the topic gathered some steam. By the end of the seventies, estimation of $P(X<Y)$ was carried out for the major distributions such as exponential (Kelly *et al.*, 1976, Tong, 1974), normal (Church and Harris, 1970, Dowton, 1973, Woodward and Kelly, 1977). Pareto (Beg and Singh, 1979) and exponential families (Tong, 1977). Also, significant advances in Bayes estimation of $P(X<Y)$ for exponentially and normally distributed X and Y were made by Enis and Geisser (1971). The other milestones of seventies are the introduction of non-parametric empirical Bayes estimation of $P(X<Y)$ (Fergusion, 1973, Hollander and Korwar, 1976) and the study of the system reliability (Bhattacharya and Johnson, 1974).

Applications

The stress strength models- initially originated from a seemingly unrelated problem of classical non-parametric test of equality of two distribution functions. It then naturally led to the expression of the type $P(X<Y)$ and later it was realized that these quantities could be fruitful for examining the probability of inequality type relation between two or more type random variables under a great variety of conditions and situations.

This naturally resulted in applications in numerous engineering probabilities under the banner of "reliability" provided that the random variables under consideration admit appropriate interpretation.

Next it became evident that practical applications are by no means confined to engineering, or to military problems. In fact, the advances in medical statistics in the last twenty years triggered numerous applications for medical-oriented problems of which the clinical trials are one of the fastest growing areas. Thereafter came application in psychology which required adjustment of the theory to accommodate categorical data. Further natural applications, especially in but not limited to medicine, involved comparison of two or more random variables representing the state of affairs in two or more situations at different time intervals.

The new frontier of potential application is the real world problems where the model cannot be viewed as consisting and involving independently identically distributed random variables and is more appropriately represented by a binary data leading to the so-called "ROC approach" with a strong dose of logistic regression.

One of the recent application is the challenging problem of estimating the unknown strength characteristic from a observable distribution of stress which leads to more interesting probabilistic and statistical problems. Another possible application still in its infancy is the relation between the stress-strength model and the quality control concepts and consequences.

It should be noted that as the sources of numerical data are becoming more widely available and statistical calculation becoming more accessible due to the rapid advances in computer technology more and more widespread applications are to be expected. The stress-strength relation is an universal flexible relation easily adaptable to various fields of human endeavor and natural phenomena. It is a powerful tool for comparing and dissecting interrelated situations.

12. Classical and Bayesian inference

For the reliability measurement, we have so far learned how to measure the reliability with various classical statistical models. The framework of the classical statistical approach is to evaluate procedures based on imagining repeated sampling from a particular model (the likelihood), which defines the probability distribution of the observed data with unknown fixed parameters. The accuracy of the procedure evaluation is relied on the precision of the parameters estimation with repeating sampling or observed data. The precision of the estimations would be doubtful because of some unavoidable factors, e.g. sampling error or insufficient data.

Unlike the classical statistical estimation, Bayesian approach to statistical design and analysis relies not only on the repeating sampling or observation data but also on the prior knowledge such as reliability engineers' experience or some prior belief about the parameter of interest. This would make Bayesian estimation an effective and practical alternative to the classical one.

As a part of the standard probabilistic reliability assessment for a system, we have estimated the unavailability of the system. This unavailability is a function of the unavailability parameters (e.g. maintenance period, failure rate, and demand failure probability) for the system component. These unavailability parameters are estimated using the available data.

The available data sometimes tend to be relatively few since the equipment failures tend to be relatively rare event; the available data sometimes tend to be relatively unreliable since the data sampling processes are somehow biased. Classical statistical methods are ill fitted for those kind situations, and possibly lead the whole reliability assessment to an unreliable solution.

Partly because of this flaw of classical statistical method, reliability engineers turn to Bayesian approaches. With Bayesian approach, *reliability participators could incorporate a wide variety of forms of* information in the estimation process. In the Bayesian methods, all the

uncertainties in the parameters due to the lack of knowledge are expressed via probability distributions. This is the major departure from the classical methods, since for classical methods all the parameters are true unknown value. There are no uncertain parameters being estimated.
The basic difference between Classical and Bayesian is that classical approach considers the parameter of a population a fixed quantity, whereas Bayesian regard the parameter of a distribution a random variable.
The most important thing in the Bayesian approach is the specification of a distribution on the parameter space, which has been named as 'prior distribution'. The specification of the prior distribution is mostly based on pragmatic grounds *i.e.* it is based on some previous experiment, investigation study or knowledge.

13. Bayesian methodology

Prior and posterior distributions: Let $X_1, X_2, ..., X_n$ be independently identically distributed random variables from a density $f(X|\theta)$, $\theta \in \Theta$, where the function $f(.|\theta)$ is assumed known except for θ. The problem is to estimate a specified function $\Phi(\theta)$. The Bayesian approach to estimate $\Phi(\theta)$ assumes the existence of a probability distribution on Θ. This probability distribution, specified by a completely known probability density function $g(\theta)$, describes the degree of belief in possible parameter values prior to an observation being made and consequently it is called a prior distribution. Thus the unknown θ may be considered as the realized values of some random variable θ whose probability density function $g(\theta)$ is known. The information of known $g(\theta)$ can be incorporated into estimation procedures by means of the posterior distribution of θ given $X_1 = x_1,\ X_2 = x_2, \ldots, X_n = x_n$.

$$\Pi(\theta|\ x_1,\ldots,x_n) = \frac{g(\theta)[\prod_{i=1}^{n} f(x_i|\theta)]}{h(x_1,\ldots,x_n)}$$

where, $h(x_1,\ldots,x_n) = \int_{\Omega} \prod_{i=1}^{n} f(x_i|\theta)\xi(\theta)\,d\theta.$

$\Pi(\theta|\ x_1,\ldots,x_n)$ may be interpreted as describing an experimenter's degree of belief in different possible values of θ after the observation $(x_1,\ldots,x_n)$ have been made, and consequently it is called the posterior distribution of θ. Thus the sample observations change a decision maker's degree of belief by changing a prior distribution into a posterior distribution.

14. Prior distributions

Mostly, we use two types of prior distributions:

(i). Proper Prior and (ii). Improper Prior distributions.

Proper prior distribution: If probability density function or probability mass function $g(\theta)$ is such that the integral or sums over its admissible range is one. *i.e.*

$$\int g(\theta)dx = 1 \text{ or } \sum g(\theta) = 1$$

then the prior distributions are called Proper prior distributions.

Improper prior distribution: Such priors arises when $g(\theta)$ is not a probability distribution in that $g(\theta) \geq 0$ but

$$\int g(\theta)dx = p \neq 1 \text{ or } \sum g(\theta) \neq 1.$$

However there may be other prior distributions as well as described below:

Non-informative prior distribution: When we are in a state of ignorance about the parameter we need to choose a prior, which will uniformly express our ignorance about the parameter. Such a prior is known as Non-informative prior. It is the prior that contains no information about θ. If the prior is non-informative, we should assign the same density to each $\theta \in \Omega$, which of course implies that prior $g(\theta)$ is uniformly given by $g(\theta) = k$, $\theta \in \Omega$.

The non informative prior often leads to a class of improper priors; improper in the sense that

$$\int_{\Omega} g(\theta)\, d\theta \neq 1$$

Jeffery's invariant prior: This prior also known as ignorance prior is again non-informative prior. Jefferys (1961) suggested to choose prior

$$g(\theta) \propto \sqrt{i(\theta)}$$

where $i(\theta) = -E\left[\frac{\partial^2}{\partial \theta^2} \log f(x|\theta)\right]$, is Fisher's information about θ contained in x.

Natural conjugate prior: We say that the family of prior distributions $\{g(\theta),\ \theta \in \Omega\}$, is a natural conjugate family if the corresponding posterior distribution belongs to the same family as $g(\theta)$.

15. Some basic definitions

Reliability function: According to Leith (1995), the reliability of a product is the measure of its ability to perform its function, when required, for a specified time, in particular environment. Reliability is defined as the probability that a system (component) will function over some time period $>t$ (Ebeling, 1997). We express this relationship mathematically as, if T denote the time to failure for a unit with probability density function (*pdf*) $f(t)$ and t is a pre-assigned time-point, then the reliability of the unit is defined as

$$R(t)=P\,[T>t]=1-F(t)=1-\int_0^t f(t)\,dt$$

Failure rate function: The rate at which failures occur in a certain time interval $[t_1, t_2]$ is called the failure rate during that interval. It is defined as the probability that a failure per unit time occurs in the interval, given that a failure has not occurred prior to t_1 the beginning of the interval. Thus the failure rate is given by

$$\lambda(t)=\frac{\int_{t_1}^{t_2} f(t)\,dt}{(t_2-t_1)\int_{t_1}^{\infty} f(t)\,dt}=\frac{\int_{t_1}^{\infty} f(t)\,dt-\int_{t_2}^{\infty} f(t)\,dt}{(t_2-t_1)\int_{t_1}^{\infty} f(t)\,dt}$$

If, $t_1=t$ and $t_2=t+\Delta t$, we get

$$\lambda(t)=\frac{R(t)-R(t+\Delta t)}{\Delta t.R(t)}$$

Hazard rate function: The hazard rate is defined as the limit of the failure rate as the length of the interval, $[t_1, t_2]$ approaches zero. Thus, it is instantaneous failure rate.

The hazard rate $h(t)$ is defined as

$$h(t)=\lim_{\Delta t\to 0}\frac{R(t)-R(t+\Delta t)}{\Delta t\,R(t)}=\frac{1}{R(t)}\left[-\frac{d}{dt}R(t)\right]$$

$$=\frac{d\ln R(t)}{dt}=\frac{f(t)}{R(t)}$$

The quantity $h(t)\,dt$ represents the probability that a device of age t will fail in the small interval of time t to $t+\Delta t$. Hazard rate thus indicates the changing rate in the aging behaviour over the life of a population of components. For example, two designs may provide the same reliability at a specified point in time, but the failure rate curves can be very different.

Mean time to failure (MTTF): The expected life, or the expected time during which an item functioning until first failure will perform successfully, is defined as

$$E(T)=\int_0^{\infty} t\,f(t)\,dt$$

where $f(t)$ is the *pdf* of T, the lifetime of an item. As the lifetime of an item has to be non-negative, we define $f(t)$ for $T \geq 0$.

System reliability: A system is a collection of components, subsystems and/or assemblies arranged to a specific design in order to achieve desired functions with acceptable performance and reliability. The types of components, their quantities, their qualities and the manner in which they are arranged within the system have a direct effect on the system's reliability. Some of these systems are discussed in the following

- **Series system:** In a series system, a failure of any component results in failure for the entire system. In most cases when considering complete systems at their basic subsystem level, it is found that these are arranged reliability-wise in a series configuration. For example, a personal computer may consist of four basic subsystems: the motherboard, the hard drive, the power supply and the processor. These are reliability-wise in series and a failure of any of these subsystems will cause a system failure. In other words, all of the units in a series system must succeed for the system to succeed.

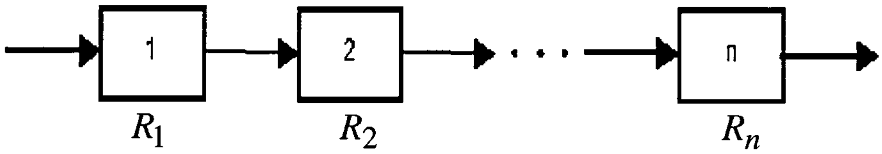

Figure 1.7: Series system

If $C_i, i = 1, 2, ..., n$ be the set of n independent components with their respective reliabilities $R_i, i = 1, 2, ..., n$, arranged in series as given in figure 1.7, then reliability of the system is

$$R_s = \prod_{i=1}^{n} R_i$$

- **Parallel system:** In a parallel system, as shown in Figure 1.8, at least one of the units must succeed for the system to succeed. Units in parallel are also referred to as redundant units. Redundancy is a very important aspect of system design and introducing redundancy is one of several methods of improving system reliability.

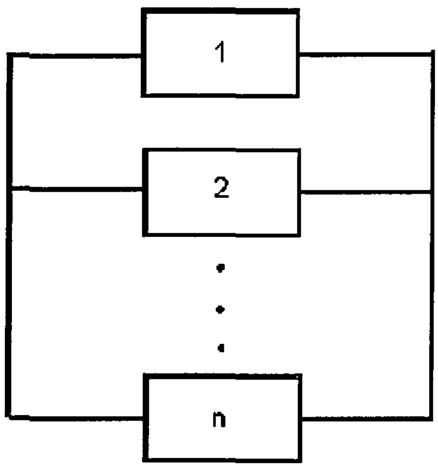

Figure 1.8: Parallel system

If C_i, $i = 1, 2, ..., n$ be the set of n independent components with their respective reliabilities R_i, $i = 1, 2, ..., n$, arranged in parallel as given in figure 1.8, then reliability of the system is

$$R_p = 1 - \prod_{i=1}^{n} (1 - R_i)$$

- $k-$**out of**$-m$ **system:** A $k-$out of$-m$ system consist of m independent and identical components and operates as long as atleast $(k \leq m)$ of its components operate. The reliability of the system with such a configuration can be evaluated using the binomial distribution

$$R_{km} = \sum_{i=k}^{m} \binom{m}{i} R^i (1-R)^{m-i}$$

In particular, for $k = m$, the system reduces to a series system and for $k = 1$, the system reduces to parallel system.

16. Some lifetime distributions

I. Exponential Distribution

A random variable T is said to have an exponential distribution if its probability density function (*pdf*) and distribution function (*df*) is of the form

$$f(t) = \frac{1}{\lambda} \exp(-t/\lambda), \quad t \geq 0, \lambda > 0$$

$$F(t) = 1 - e^{-x/\lambda}, \quad t \geq 0.$$

The reliability function and the hazard rate are

$$R(t) = e^{-\lambda t} \text{ and } h(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

The mean and variance are

$$E(T) = \frac{1}{\lambda} \text{ and } V(t) = \frac{1}{\lambda^2}.$$

The constant failure rate λ can be interpreted to mean that the failure process has no memory. Then using the law of conditional probability, means that

$$P(t \le T \le t+\Delta t | T > t) = \frac{R(t) - R(t+\Delta t)}{R(t)} = 1 - e^{-\lambda.\Delta t}$$

which is independent of t. Thus, if the device is still functioning at time t, it is as good as new.

Exponential distribution is used commonly in reliability engineering due to its simplicity and also due to fact that most of the lifetime distributions conform to exponential law. Davis (1952), Epstein (1958) and Barlow and Proschan (1965) are among those who have put forth arguments in its favour.

II. Rayleigh distribution

A random variable X is said to have the Rayleigh distribution if its *pdf* $f(x)$ is given by

$$f(x) = \frac{2x}{\theta} e^{-x^2/\theta}, \quad x, \theta > 0$$

If we replace θ by $2\lambda^2$, then the *pdf* is given by

$$f(x) = \frac{x}{\lambda^2} e^{-x^2/2\lambda^2}, \quad x, \lambda > 0$$

and the *df* is given by

$$F(x) = 1 - e^{-x^2/2\lambda^2}, \quad x > 0 \text{ or } F(x) = 1 - \lambda\sqrt{2\pi}\ \phi(x/\lambda), \quad x > 0$$

where $\phi(.)$ is the *pdf* of standard normal variate and the mean of the distribution is $E(X) = \lambda\sqrt{\frac{\pi}{2}}$.

Another possible variation of the form of the Rayleigh distribution is given as

$$f(x) = \frac{\pi x}{2\beta^2} \exp\left[-\frac{\pi x}{4\beta^2}\right], \quad x, \beta > 0$$

This density function has the advantage that $E(X)=\beta$, i.e. β represents the population mean. Rayleigh distribution is positive skew and kurtosis is also close to normal. The distribution is unimodal with mode at β. Siddique (1962) and Archer (1967) gave the useful summary of its properties. Polovko (1968) pointed the importance of the distribution in electro-vacuum devices and communication engineering. It has characteristic property of having a failure rate proportional to lifetime elapsed.

III. Normal Distribution

A random variable T is said to have the normal distribution with parameters μ and σ^2, if its *pdf* is given by

$$f(t)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}, \quad -\infty<t<\infty, \ \mu,\sigma^2>0$$

or $$f(t)=\phi(\frac{t-\mu}{\sigma})\Big/\sigma$$

and the *df* is given by

$$F(t)=\int_{-\infty}^{t}\frac{1}{\sigma\sqrt{2\pi}}\exp[-\frac{1}{2}(\frac{x-\mu}{\sigma})^2]dx=\Phi(\frac{t-\mu}{\sigma})$$

where $\Phi(t)=\int_{-\infty}^{t}\phi(x)dx$ and $\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

Therefore, reliability and the hazard rate functions are

$$R(t)=1-\Phi(\frac{t-\mu}{\sigma}) \text{ and } h(t)=\frac{\phi(\frac{t-\mu}{\sigma})}{\sigma[1-\Phi(\frac{t-\mu}{\sigma})]}.$$

The mean of normal distribution is $E(T)=\mu$ and variance is σ^2.

IV. Weibull Distribution

The Weibull distribution is most commonly used probability distribution in the field of failure data analysis. The distribution is named after Weibull (1939) who used it to represent the distribution of breaking strength of materials. It has also been used to describe vacuum-tube failures (Kao, 1959) and ball-bearing failures (Lieblein and Zelen, 1956). The Weibull distribution encompasses both increasing and decreasing hazard rates, and has been used to describe both initial failures as well as, when a system is composed of a number of components, the failures due to the most severe defect of a large number of possible defects (Von Alven, 1964).

The general form of the *pdf* of Weibull distribution is

$$f(t)=\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^{\beta}}, \quad \alpha, \beta, t \geq 0.$$

where β is the shape parameter and α is the scale parameter.

If $\beta=1$, the distribution becomes the exponential distribution and for $\beta=2$, it becomes the Rayleigh distribution.

The reliability and the hazard rate functions are

$$R(t)=e^{-(t/\alpha)^{\beta}} \text{ and } h(t)=\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}, \quad \beta,\alpha>0,\ t\geq 0.$$

The mean and variance of Weibull distribution are

$$E(T)=\alpha\,\Gamma\left((\beta+1)/\beta\right) \text{ and } V(T)=\alpha^{2}\left[\Gamma\left(\frac{\beta+2}{\beta}\right)-\Gamma^{2}\left(\frac{\beta+1}{\beta}\right)\right].$$

Another form, in which Weibull distribution has been used in our study, is

$$f(x)=\frac{\lambda x^{\lambda-1}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}}{\beta^{\lambda}} e^{-(\frac{x}{\beta})^{\lambda}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}}, \quad x,\lambda,\beta>0$$

It has the advantage that $E(X)=\beta$.

Berrettoni (1964) has described many applications of the Weibull distribution using graphical methods. It is sometimes used as a tolerance distribution in the analysis of quantal response data. Many applications found in papers by Freudenthal and Gumbel (1954), Plait (1962), Johnson (1968) and Jaech (1968). Haq and Khan (1987) describe the method of structural estimation of the shape parameter of Weibull distribution.

V. Gamma distribution

A random variable X is said to have a gamma distribution if its *pdf* is given by

$$f(x)=\frac{(x/\theta)^{\beta-1}e^{-(x/\theta)}}{\theta\,\Gamma\beta},\quad x\geq 0,\beta,\theta>0.$$

where β is the shape parameter, θ is scale parameter and Γ is the gamma function defined as

$$\Gamma(p)=\int_0^\infty x^{p-1}e^{-x}\,dt$$

and the corresponding *df* is given by

$$F(x)=\frac{\Gamma_t(\beta)}{\Gamma(\beta)},\quad x\geq 0,\ \beta>0$$

where $\Gamma_t(p)$ is an incomplete gamma function defined as

$$\Gamma(p)=\int_0^p x^{p-1}e^{-x}\,dt.$$

The values of the above incomplete gamma function have been approximated for given values of p, by Pearson (1957). Otherwise the same is obtained by using appropriate Statistical Package.

VI. Beta Distribution

The lifetime distributions considered so far have one characteristic in common, that is, the probability density function of each one of them except the normal distribution is defined over $(0, \infty)$. This is equivalent to saying that lifetime of an equipment can be as large as possible, at least theoretically. In practice, there is always an upper limit of time for which the functioning of an item is required. Keeping this in mind we look for a lifetime distribution that is defined over a finite time period. One such family of distribution is given by the probability density function

$$f(x) = \frac{1}{B(a,b)} \frac{(x-a)^{p-1}(b-x)^{q-1}}{(b-a)^{p+q-1}}$$

where $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$ is known as Beta function and the family of distributions is called Beta-family of distribution.

VII. Power function distribution

A random variable X is said to have a power function distribution if its *pdf* and *df* are of the form given below

$$f(x) = \left(\frac{a}{\theta}\right)\left(\frac{x}{\theta}\right)^{a-1}, \quad 0 < x \le \theta, \quad a, \theta > 0$$

$$F(x) = \left(\frac{x}{\theta}\right)^{a}, \quad a, \theta > 0$$

It can be seen that this is a member of Beta family of distributions.

Mean and variance are, $E(X) = \frac{a\theta}{a+1}$ and $V(X) = \frac{a\theta^2}{(a+1)^2(a+2)}$.

Reliability, hazard rate function and coefficient of variation are

$$R(x)=1-(x/\theta)^a,\ \ h(t)=\frac{a\,x^{a-1}}{\theta^a-x^a}\ \text{and}\ \ CV=\frac{1}{\sqrt{a(a+2)}}.$$

Of course we have introduced the scale parameter θ. The scale parameter θ represents the maximum time up to which the equipment is supposed to work. Without loss of generality, we may fix $\theta=1$.

VIII. α-Distribution

Gertsbaph and Kordousky (1969) and Vysokovskii (1970) developed the Berstein probability density function as a result of wear analysis of broad nosed cutting tools. Vysokovskii renamed the Berstein probability density function as the α-distribution, which has further been discussed by Katsev (1968), Pronkov (1973), Kendall and Sheikh (1979), Pandit and Sheikh (1980) and Wager and Barash (1971). Sheriff (1983) and Ahmad and Sheikh (1981, 1984) proposed to use the α-probability density function in modelling lifetimes under accelerated test conditions and compared with many standard density functions.

For half alpha distribution for values of α and c, we have

$$f(x)=\frac{1}{\Phi(1/\sqrt{\alpha})}\frac{c}{\sqrt{2\pi\alpha}}\frac{1}{x^2}\exp\left[-\frac{1}{2\alpha}\left(1-\frac{c}{x}\right)^2\right],\ \ x>0,\ \alpha,c>0.$$

the *pdf* is unimodal with mode at $x=\dfrac{2c}{[1+\sqrt{1+8\alpha}\,]}$.

The α-distribution has also been developed to model the life characteristics of machine components, which deteriorate according to a scheme of non-stationary linear random wear process. The α model has been successfully used in a variety of situations such as,

(i). Modelling the cutting tool life

(ii). Monitoring the dimension of machine parts for statistical quality control

(iii). Size modelling (Ahmad and Chaudhary, 1992)

IX. Geometric Distribution

The geometric distribution $P\,(X = x) = pq^x$, $x = 0, 1, 2, \ldots$ and $q = (1 - p)$ has been suggested to study the number of successful cycles and has been interpreted as the probability of x success followed by 1 failure.

For geometric distribution

$$P\,[X = x + k \mid X \geq x] = P[X = k]$$

which says that even though it has tested x cycles without a failure, the probability of testing another k cycles without a failure is same as it was in the beginning, that is to say, it is independent of time. Therefore, it is a discrete analogue of continuous exponential distribution.

Or, A series of independent identical trials is performed. Each trial can either succeed or fail, and the trials are repeated until the first success. The parameter p represents the probability of success on a single trial and the random variable X represents the number of trials performed. $X \in (1, 2, 3, \ldots)$. The probability mass function *(pmf)* for X is given by

$$P(x) = p\,(1 - p)^{x-1}, \quad x = 1, 2, 3, \ldots$$

and the cumulative distribution function (*cdf*) for X is given by

$$F\,(x) = \sum_{i=1}^{x} p\,(1 - p)^{i-1}, \quad x = 1, 2, 3, \ldots$$

The mean and Variance are

$$E(X) = \frac{1}{p} \text{ and } V\,(X) = \frac{1 - p}{p^2}.$$

X. Negative binomial distribution

A series of independent and identical trials are performed. Each trial can either succeed or fail, and the trials are repeated until k successes occur. The parameter p represents the probability of success on a single trial and the random variable X represents the number of trials performed. $X \in \{k, k+1, k+2, \ldots\}$

The *pmf* for X is given by

$$p(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}, \qquad x = k, k+1, k+2, \ldots$$

and the *cdf* for X are given by

$$F(x) = \sum_{i=k}^{x} \binom{i-1}{k-1} p^k (1-p)^{i-k}, \quad x = k, k+1, k+2, \ldots$$

The mean and variance of X is given by

$$E(X) = \frac{k}{p} \quad \text{and} \quad V(X) = \frac{k(1-p)}{p^2}$$

If we put $k = 1$, the distribution reduces to geometric distribution.

*ON FACING RAYLEIGH STRESS WITH STRENGTH HAVING POWER FUNCTION DISTRIBUTION

1. Introduction

In practical applications, reliability of an engineering product is defined as the probability that the produced item will perform a required function, under stated conditions, for a stated period of time. Sudden failure of the item usually occurs when the combined effect of the stresses imposed on it exceeds its ability to perform the required function. This is a standard definition of item failure and usually assumed to imply a catastrophic failure event. However, it is a part's ability to perform its required function under normal conditions, rather than its total (i.e. catastrophic) failure, that denotes the reliability of the item.

Extrapolating to system-level reliability, the definition of failure does not only include a catastrophic event, but it includes the degradation of one or more devices to a level that a system cannot perform a required function within useable limits. Thus, a system may fail as its devices degrade and before those devices reach catastrophic failure. It always shows that a typical manufactured part is subjected to three rates of failure. Failure rate is high initially because weaker devices fail early. This early failure is referred to as 'infant mortality' and it occurs because of an aberrant manufacturing process or an application of stress that exceeds a device's physical strength. Infant mortality may be overcome by rigorous monitoring and control of the manufacturing process, and by applying less than the rated maximum stress on the part. As components begin

* Parts of the results of this chapter appeared in Khan and Islam (2007a).

their usable lifetime, the failure rate becomes relatively constant. Throughout this period, failures are generally a result of random device-overload. Finally, wear-out occurs as devices reach end-of-life - where the combined effect of the stresses exceeds the strength of the part – and the failure rate increases significantly.

The basic impetus to those developments can perhaps be ascribed to the specific practical problem of applied statistics encapsulated by the term *stress-strength. In the simplest term this can be described as an* assessment of reliability of an item or component of a system, in terms of random variable X representing stress experienced by the item (or component) and Y representing the strength of the item (or component) available to overcome the stress. According to this simplified scenario if the stress exceeds the strength $(X > Y)$ the component would fail; and visa versa. Reliability is then defined as the probability of not failing. To distinguish it from the reliability of an item functioning until first failure, we call it 'strength-reliability' of the item and denote it by $R = P\,(Y > X)$. In order to make the concept clear, let us consider the following examples:

- The receptor of a communication system operates only if it is stimulated by a source whose random magnitude Y is greater than a random lower threshold X for the system. Here, R is obviously the probability that the receptor operates.

- If X represents the maximum chamber pressure generated by ignition of a solid propellant and Y represents the strength of the rocket chamber, then R is the probability of successful firing of the engine.

- If X represents the diameter of a shaft and Y represents the diameter of a bearing that is to be mounted on the shaft, then R is the probability that the bearing fits without interference.

In many applications, the reliability has to be very close to '1' for the device to have any possibility of useful life. One consequence is that very large samples may be needed to obtain sufficiently accurate estimates of the reliability since we are here dealing with extreme tails of distribution. A further and even more serious difficulty lies in the sensitivity of reliability to small changes from assumed models for the distribution of X and Y.

In the words of Harris and Soms (1983):
"relatively small perturbations of the tail of the strength distribution can make the failure probability far higher than may be desirable, particularly, where failure can be catastrophic". This can lead to the cases where "the estimation procedures produced results which were significantly contradicted by subsequent experience".

The germ of the idea was introduced by Birnboum (1956) and developed by Birnboum and McCarty (1958). The formal term "Stress-Strength" appears in the title of Church and Harris (1970), providing an example of a missile flight, where the initial values of the stress correspond to propulsive force, angles of elevation and atmospheric conditions etc. Later, Dowton (1973), Beg and Singh (1979), Reiser and Guttman (1986), and Nandi and Aich (1994) studied the problem about the evaluation of strength reliability i.e. $P(Y > X)$, (where X and Y stand for stress and strength respectively and are assumed to follow some known form of probability distributions) and discussed its statistical properties.

Recently Alam and Roohi (2002) have studied the problem of Stress-Strength reliability in a different perspective. In addition to finding $P(Y > X)$ for a given set of distributions, they have found the required parametric values of the assumed distribution so that a desired level of strength reliability may be achieved. They have assumed exponential strength and exponential stress for this purpose. Alam and Roohi (2003) again discussed the problem of Stress-Strength considering exponential stress with strength having power function distribution. In this chapter, we study the problem of Stress-Strength considering Rayleigh stress with strength having power function distribution.

As pointed out by Alam and Roohi (2003), choice of stress distribution with an infinite range is justified, as it is genuinely possible to face a very huge stress that may be regarded as tending to infinity. This observation motivates the choice of Rayleigh distribution, instead of exponential. In addition it also closely resembles to Normal distribution for small values and very small probabilities for large values, as it should be. Figure 2.1 gives the graphic representations of the probability density function of Rayleigh distribution in its simplest form:

$$f(x) = \frac{x}{\beta^2} \exp\left[-\frac{1}{2} \left(\frac{x}{\beta} \right)^2 \right], \; x, \beta > 0 \tag{1.1}$$

for selected values of β.

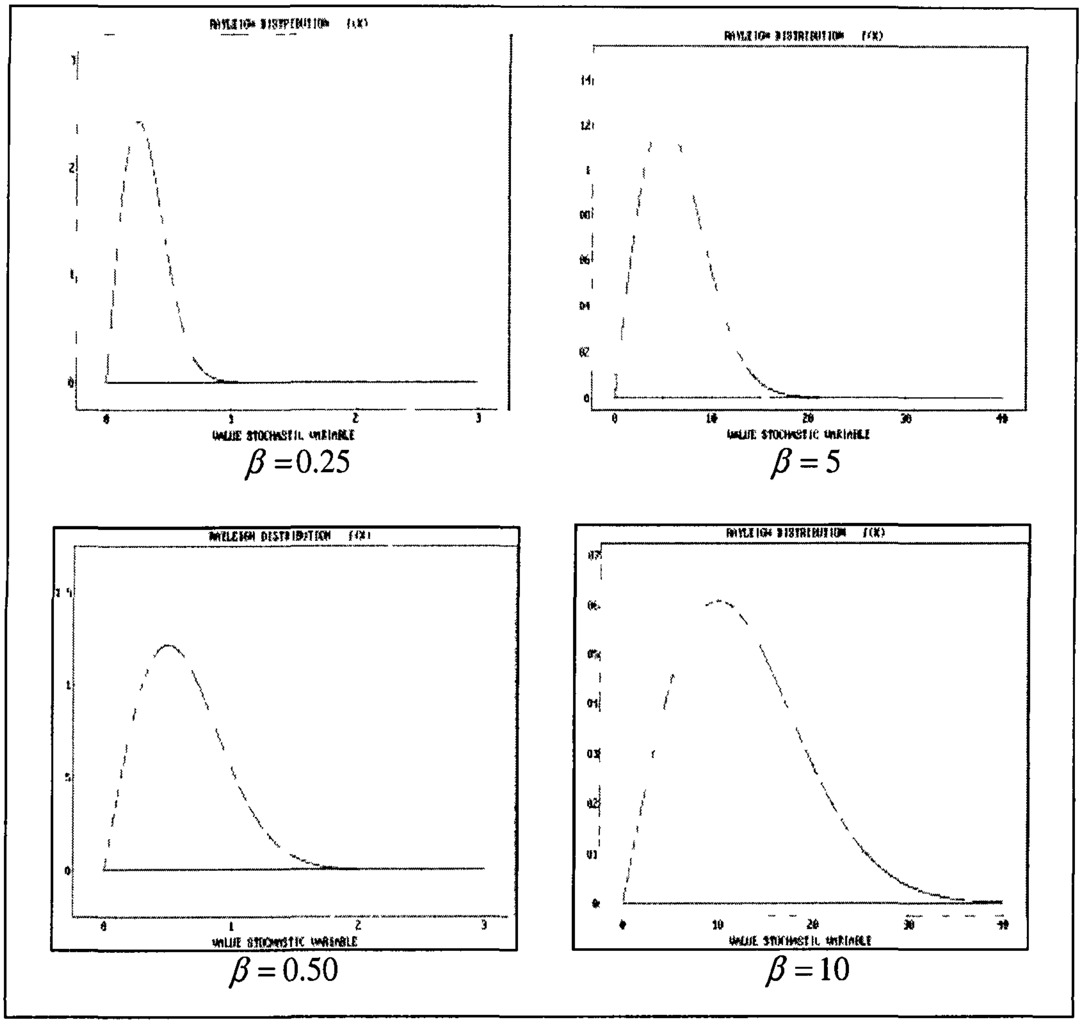

Figure 2.1: Rayleigh probability density function

The choice of Power function distribution as strength distribution emanates from the fact that the designed strength of equipment should only be limited to finite range. This is so because the strength of an engineering product is always a function of a combined strength of set of subcomponents and as we know that the strength of a chain lies in its weakest link, not all the subcomponents are likely to have an infinite strength. As such we assume that the strength follows a power function distribution defined over a finite range 0 to θ, having 'θ' as a scale parameter and 'a' as the shape parameter. It can be shown that coefficient of variation of power function is $\frac{1}{\sqrt{a(a+2)}}$, so that the probability of the strength remaining concentrated around its mean value increases with an increase in the value of 'a'.

2. Derivation of main results

In what follows, X represents the stress faced by an item and Y stands for its strength, both being random variables.

Let X and Y have the probability density functions $f(x)$ and $g(y)$, given by

$$f(x)=\frac{\pi x}{2\beta^2}\, e^{-\left(\frac{\pi x^2}{4\beta^2}\right)}, \quad \beta>0 \tag{2.1}$$

$$g(y)=\left(\frac{a}{\theta}\right)\left(\frac{y}{\theta}\right)^{a-1}, \quad 0<y<\theta, a>0 \tag{2.2}$$

2.1 Meeting a disaster or catastrophic situation

Since the maximum possible value of Y is θ, Y cannot exceed X if X exceeds θ. Alam and Roohi (2003) have rightly described such a situation, as a 'disaster'. We prefer the word 'catastrophe' to describe such a situation. In practice, one cannot avert a 'disaster' or a 'catastrophe'. However, if possible, it is desirable to minimize the chances of facing such a scenario. We start with finding the probability of a catastrophic situation i.e. $P(X>\theta)$.

By definition:

$$P[X>\theta]=\int_{\theta}^{\infty}\frac{\pi x}{2\beta^2}\, e^{-\left(\frac{\pi x^2}{4\beta^2}\right)}dx$$

$$= e^{-\frac{\pi\theta^2}{4\beta^2}}$$

For a fixed θ and known β, if we, let $k = \frac{\theta}{\beta}$, then

$$P[X > \theta] = e^{-\frac{\pi}{4}k^2} = e^{-.786k^2} \tag{2.3}$$

Table 2.1 shows the probability of a catastrophe for selected values of k.

Table 2.1: $P[X > \theta]$, the probability of catastrophe

k	$P[X > \theta]$
0.25	0.9520
0.50	0.8216
0.75	0.6426
1.0	0.4556
1.25	0.2928
1.50	0.1706
2.0	0.0431
2.5	0.00735
3.0	0.000846
3.5	0.0000658
4.0	0.00000345
5.0	0.00000000292

Following points are to be noted here:

(i). Chances of a catastrophic situation decrease as the value of k increases. Clearly, this implies the increase in the maximum possible strength θ with respect to average stress β.

(ii). Chances of a catastrophic situation do not vanish, howsoever large the maximum possible strength θ may be. However, we may find the suitable value of k so that probability of a catastrophic situation may be a pre-determined value of α that may be regarded as the tolerance level of the item. Naturally, we would like this α to be as small as possible. Clearly, (2.3) gives

$$\alpha = P[X > \theta] = e^{-.786 k^2}$$

which, in turn, implies that for a fixed value of α, k is given by

$$k = \sqrt{\frac{-\ln \alpha}{0.786}} \tag{2.4}$$

Table 2.2 gives the values of k for selected tolerance levels α.

Table 2.2: values of k for selected tolerance levels α

α	0.1	0.05	0.02	0.01	0.001	0.0001	0.00001
k	1.7115	1.9522	2.2309	2.4205	2.9645	3.4231	3.8272

2.2 Stress exceeding certain level of strength

If strength follows probability density function given by (2.2), the average strength is $\left(\frac{a\theta}{a+1}\right)$. Hence the probability that stress will exceed average strength is given by

$$P[X > E(Y)] \equiv P[X > (\frac{a\theta}{a+1})] = \int_{\frac{a\theta}{a+1}}^{\infty} \frac{\pi x}{2\beta^2} e^{-\pi x^2/4\beta^2}\, dx$$

Let $\frac{\pi x^2}{4\beta^2} = t$, so that $\frac{\pi x}{2\beta^2} dx = dt$, we get

$$P[X > E(Y)] = e^{-\frac{\pi}{4\beta^2}\left(\frac{a\theta}{a+1}\right)^2} \tag{2.5}$$

For a fixed θ and known β, if we define $l = \frac{a\theta}{\beta(a+1)}$, then

$$P[X > \left(\frac{a\theta}{a+1}\right)] = e^{-.786 l^2} \tag{2.6}$$

If we fix this probability as α_1, then

$$l = \sqrt{\frac{-\ln \alpha_1}{0.786}} \tag{2.7}$$

The above result can be generalized as follows:

For some constant c

$$P[X > (c\theta)] = \int_{c\theta}^{\infty} \frac{\pi x}{2\beta^2} e^{-\pi x^2/4\beta^2} dx$$

$$= e^{-\frac{\pi}{4\beta^2}(c\theta)^2} \tag{2.8}$$

For a fixed θ and known β, if we let $m = \frac{c\theta}{\beta}$, then

$$P[X > (c\theta)] = e^{-.786 m^2} \tag{2.9}$$

If we fix this probability as α_2, then

$$m = \sqrt{\frac{-\ln \alpha_2}{0.786}} \tag{2.10}$$

It may be noted here that if we let $c = 1$, then $m = k$ and (2.9) reduces to (2.3). Further, if $c = \frac{a}{a+1}$, then $m = l$ and (2.9) reduces to (2.6).

One may wonder here about the desirability of having (2.3) and (2.6) while these are the special cases of (2.8). We shall discuss their utility in section 3.

2.3 Strength reliability with respect to stress

Our ultimate aim here is to find the reliability of the item under stress and strength having the statistical formulation given at the beginning of section 2, expressed through (2.1) and (2.2).

Using the result given in Nandi and Aich (1994), the strength reliability of an item is given by

$$R = P\,[Y > X\,] = \int_0^{\infty} x\, f\,(x)\, \{ \int_1^{\infty} g\,(vx)\, dv \ \}\ dx$$

In our case

$$R = \int_0^{k\beta} \int_1^{k\beta/x} x \frac{\pi\, x}{2\beta^2} e^{-\frac{\pi\, x^2}{4\beta^2}} \frac{a}{(k\,\beta)^a} (vx)^{a-1}\, dv\ dx$$

where $k = \dfrac{\theta}{\beta}$. It can be easily seen that

$$R = \frac{\pi}{2\beta^2} \frac{a}{(k\beta)^a} \int_0^{k\beta} x^{a+1} e^{-\frac{\pi\, x^2}{4\beta^2}} \left\{ \int_1^{k\beta/x} (v)^{a-1}\, dv \right\} dx$$

$$= \frac{\pi}{2\beta^2} \frac{a}{(k\beta)^a} \int_0^{k\beta} x^{a+1} e^{-\frac{\pi\, x^2}{4\beta^2}} \left\{ \left(\frac{k\,\beta}{x} \right)^a - 1 \right\} dx$$

$$= \frac{\pi}{2\beta^2} \int_0^{k\beta} x\ e^{-\frac{\pi\, x^2}{4\beta^2}}\ dx - \frac{\pi}{2\beta^2} \frac{1}{(k\beta)^a} \int_0^{k\beta} x^{a+1}\ e^{-\frac{\pi\, x^2}{4\beta^2}}\ dx$$

$$= [1 - e^{-\frac{\pi}{4}k^2}\,] - \frac{1}{\left(\frac{\pi}{4} k^2 \right)^{a/2}} \int_0^{\frac{\pi}{4}k^2} t^{a/2}\, e^{-t}\, dt, \text{ where } t = \frac{\pi\, x^2}{4\beta^2}$$

The above expression is complicated and it is not easy to evaluate R even for given values of a and k. However, it can be approximated by using table of incomplete Gamma function, provided by Pearson (1957) or some appropriate Statistical Package. We obtain $R = P\,[Y > X\,]$ for selected values of a and k as shown in Table 2.3.

Table 2.3: $P[Y > X]$, **Strength exceeds Stress**

$k \rightarrow$ $a \downarrow$	1.0	1.5	2.0	2.5	2.75	3.0	3.5	4.0	4.5
1.0	0.2337	0.3829	0.5045	0.6009	0.6364	0.6668	0.7144	0.7500	0.7778
2.0	0.2787	0.5192	0.6970	0.7976	0.8321	0.8587	0.8961	0.9204	0.9371
3.0	0.3602	0.6183	0.7842	0.8804	0.9087	0.9295	0.9555	0.9701	0.9790
4.0	0.3585	0.6687	0.8349	0.9211	0.9444	0.9602	0.9784	0.9873	0.9920
5.0	0.4377	0.7086	0.8617	0.9432	0.9626	0.9753	0.9884	0.9940	0.9967
6.0	0.4379	0.7114	0.8801	0.9554	0.9730	0.9835	0.9933	0.9969	0.9985
7.0	0.4384	0.7449	0.8942	0.9647	0.9795	0.9883	0.9961	0.9983	0.9992
8.0	0.4389	0.7720	0.9060	0.9701	0.9834	0.9909	0.9972	0.9990	0.9996
9.0	0.4393	0.7748	0.9166	0.9741	0.9867	0.9930	0.9980	0.9994	0.9997
10.0	0.4398	0.7757	0.9169	0.9772	0.9886	0.9943	0.9986	0.9996	0.9998

3. Discussions

Having obtained mathematical results in section 2, we now proceed to discuss their practical utility, assuming that the strength of the item under consideration follows power function distribution given by (2.2). Further, not only the values of the parameter a and θ are either known or may be approximated through usual statistical techniques, we may also re-design the equipment with required values of these parameters (atleast approximately, with a certain degree of precision). As regards the stress, it is supposed to be beyond our control, but its probability density function may be approximated through a proper choice of β.

(i). For a fixed θ and known β, (2.3) gives the probability of facing a catastrophic situation. Going through Table 2.1, we notice that probability of facing such a situation is less than .05 if $k = 2$, i.e. θ is twice the average stress β. If we can re-design the equipment so that $k = 2.5$, i.e. θ become 2.5 times β, then the probability of facing a catastrophe is less than 0.01 and for $k=3$ it is less than 0.001. Of course, cost considerations are also to be taken care of and one may wish to think of ways to reduce the values of β. In practice, it is advisable to fix the tolerance α and use (2.4) to determine k and then think of the ways of increasing θ and reducing β simultaneously so as to achieve the required value of k. For example, suppose $\theta=100$ and $\beta=50$. If we fix $\alpha=0.001$, we find desired value of k to be 2.96 whereas the observed value is 2. Let us think of increasing θ to 120 and reducing β to 40.

(ii). (2.6) is similar to (2.3) except that θ in (2.3) has been replaced by $\left(\frac{a\theta}{a+1}\right)$ in (2.6), which is the average strength. Obviously, the desired probability of meeting the average stress can be computed as a function of a, θ and β. In order to obtain this desired probability, for a known β, we must adjust both a and θ. Obviously, in order to increase average strength, one must try to increase both a and θ. This gives us the flexibility in working out the new design; if one cannot be increased beyond a limit, try increasing the other. For example, if $\theta=100$, $a=1$ and $\beta=50$, the value of l in (2.6) is '1' and hence the probability of the stress exceeding average strength is given by $e^{-0.786}=0.4556$. Suppose θ cannot be extended beyond 120 and a can be increased up to 2, then for β remaining unchanged, probability of stress exceeding average strength reduces to 0.1337.

(iii). Table 2.3 can be used to find strength reliability of an item with respect to the stress as defined by (2.1) and (2.2). Also, it may be used to find desirable values of a and θ for a known β, in order to have pre-determined strength reliability. For example, suppose $\beta=100$. Strength reliability of 0.995 or more can be obtained for various combination of a

and k. Keeping in mind the possible extension of a and θ, we may determine the optimal combination.

(iv). A comparison with the results obtained by Alam and Roohi (2003) shows that an item whose strength follows a Power function distribution is better suited to face a stress having Rayleigh distribution as compared to the stress following an exponential distribution.

*ON WEIBULL STRESS WITH STRENGTH HAVING POWER FUNCTION DISTRIBUTION

1. Introduction

In this chapter, problem of strength of a manufactured item with power function distribution facing stress that follows a Weibull probability density function has been considered and its particular cases are discussed.

Quality, a desirable characteristic that a product or service should posses, is a key factor leading to business success, growth, and an enhanced competitive position. Because of today's competitive market, it is often not only desirable but also necessary to maximize the reliability of a product to ensure customer satisfaction and product success. The challenge that business faces is not only to develop products that are reliable but also to take into consideration the cost factors. Achieving high levels of reliability while minimizing cost often poses problems and limitations for engineers during the design stage. Therefore, a cost-reliability compromise will always exist in the context of system design.

In today's technological world nearly everyone depends upon the continued functioning of a wide array of a complex machinery and equipment for their everyday health, safety, and also for computers, electrical appliances, light, television, etc. to function whenever we need them-day after day, year after year. When they fail the results can be catastrophic: injury, loss of life and/or costly lawsuits can occur. More often, repeated customer dissatisfaction that can play havoc with the

* Parts of the results of this chapter appeared in Khan and Islam (2007b).

responsible company's market place positions. It takes a long time for a company to build up reputation for reliability, and only a short time to be branded as "unreliable" after shipping a flawed product. Continual assessment of new product reliability and ongoing control of the reliability of everything shipped are critical necessities in today's competitive business arena.

A primary goal of reliability and design engineers is to choose the best *structural and mechanical designs, considering factors such as cost,* reliability, weight and volume. The question then arises as to how to incorporate these factors into a model that will achieve optimum results. Because the factors that influence the failure of a product are often probabilistic in nature, it is important to incorporate the randomness of the design variables into a model that optimizes the final product. However, this randomness motivates some designers to believe that component failure may be entirely eliminated by using a preconceived margin as a safety factor. The reliability of a component is an important factor that needs to be considered at earlier stages of design. It has been proven that conventional design methodologies may not be adequate from a reliability point of view. A new probabilistic design methodology has been introduced and it explicitly identifies all the important design parameters and variables. The two important random variables that have been considered are stress and strength. Hence, determining the probability distributions for these variables is a key step in calculating component reliability. For a certain mode of failure, the reliability of a component with respect to the particular mode of failure is the probability that the strength of the component is greater than the stress acting on the component.

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Stress and strength are time varying in many real life systems but typical statistical models for stress-strength systems are static. A stress-strength system fails as soon as the applied stress X is at least as much as the strength Y of the system. This problem arises in the classical stress-strength reliability where one is interested in assessing the proportion of the times the random strength Y of a component exceeds the random stress X to which the component is subjected. If $(Y \leq X)$, then either the component fails or the system that uses the component may malfunction. We call $P(Y > X)$ the 'Strength-reliability' of an item and denote it by R *i.e.* $R = P(Y > X)$. This problem also arises in situations where X and Y represent the lifetime of two devices and one wants to estimate the probability that one fails before the other. Some practical examples can be found in Hall (1984) and Weerahandi and Johnson (1992). Hall (1984) provided an example of system application where the breakdown voltage Y of a capacitor must exceeds the voltage output X of a transverter (power supply) in order for a component to work properly. Weerahandi and Johnson (1992) presented a rocket-motor experiment data where Y represents the chamber burst strength and X represent the operating pressure.

The choice of Weibull stress is made because as and when shape parameter becomes large the density gets more peaked and symmetric around its mean. Other advantages of this choice include the fact that the exponential and Rayleigh distributions are the particular cases of Weibull distribution for different values of the shape parameter. As such the results obtained by Alam and Roohi (2003) and Khan and Islam (2007a) are the particular cases of the results obtained in this chapter. In addition the main feature of this density function is that it provides the theoretical U-shaped curve for the failure rate, which contains the initial failure, chance failure and wear out failure. It is very important because the lifetime of electronic, electromechanical and mechanical products are often modeled with this feature. In survival analysis, the lifetime of human beings exhibit this pattern.

As regards the choice of power function distribution as strength distribution, we refer to the arguments given in chapter II (pp-44).

2. Computation of reliability

Let X represents the stress faced by an item and Y stands for its strength, both being random variables, and let X and Y have the probability density function $f(x)$ and $g(y)$, respectively given by

$$f(x)=\frac{\lambda x^{\lambda-1}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}}{\beta^{\lambda}} e^{-(\frac{x}{\beta})^{\lambda}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}} , \quad x,\lambda,\beta>0 \quad (2.1)$$

where λ is the shape parameter and β is scale parameter, and

$$g(y)=\left(\frac{p}{\theta}\right)\left(\frac{y}{\theta}\right)^{p-1} , \quad 0<y<\theta, \; p>0 \qquad (2.2)$$

2.1 Facing a disaster situation

Since the maximum possible value of Y is θ, Y cannot exceed X if X exceeds θ. In practice, one cannot avert a 'disaster'. However, if possible, it is desirable to minimize the chances of facing such a scenario. To this effect, we find $P(X>\theta)$ as it shows the unreliability of the strength Y against the stress X. Straightforward computation shows that

$$P(X>\theta)= e^{-[\frac{\theta}{\beta}.\Gamma(\frac{1}{\lambda}+1)]^{\lambda}} .$$

For a fixed θ and known β, let $m=\frac{\theta}{\beta}$, then

$$\alpha=P(X>\theta)= e^{-[m.\Gamma(\frac{1}{\lambda}+1)]^{\lambda}}$$

For shape parameter $\lambda = 1$, this reduces to

$$\alpha_1 = P(X > \theta) = e^{-m_1} \tag{2.3}$$

as obtained by Alam and Roohi, (2003) for the exponential stress case.
Similarly, for shape parameter $\lambda = 2$, it gives

$$\alpha_2 = P(X > \theta) = e^{-0.786 m_2^2} \tag{2.4}$$

which is the result obtained by Khan and Islam, (2007a) for the Rayleigh stress case.
For shape parameter $\lambda = 3$, we get

$$\begin{aligned} \alpha_3 = P(X > \theta) &= e^{-[m.\Gamma(\frac{1}{3}+1)]^3} \\ &= e^{-0.712 m_3^3} \end{aligned} \tag{2.5}$$

and, for shape parameter $\lambda = 4$, we get

$$\begin{aligned} \alpha_4 = P(X > \theta) &= e^{-[m.\Gamma(\frac{1}{4}+1)]^4} \\ &= e^{-0.675 m_4^4} \end{aligned} \tag{2.6}$$

Table 3.1 shows the probability of a 'disaster' for $\lambda = 3$ and 4 respectively, for selected values of m.

Table 3.1: $P[X > \theta]$**, the probability of disasters for** $\lambda = 3$ **and** 4.

$m_3 = m_4 = m$	$\alpha_3 = P[X > \theta]$	$\alpha_4 = P[X > \theta]$
	$\lambda = 3$	$\lambda = 4$
0.1	0.99928	0.99993
0.2	0.99432	0.99892
0.25	0.98893	0.99736
0.50	0.91484	0.95869
0.75	0.74054	0.80769
1.00	0.49066	0.50915
1.25	0.24891	0.19244

1.50	0.09044	0.03280
1.75	0.017949	0.11273
2.0	0.0033593	0.00002039
2.25	3.004×10^{-5}	3.068×10^{-8}
2.50	1.473×10^{-5}	3.538×10^{-12}
2.75	3.140×10^{-7}	1.715×10^{-17}
3.00	4.478×10^{-9}	1.798×10^{-24}
3.25	2.427×10^{-11}	1.969×10^{-33}
3.50	5.524×10^{-14}	1.022×10^{-44}

Following points are noted here:

a). Chances of a disaster decrease as the value of m increases. i.e., increase in maximum possible strength θ with respect to average stress β.

b). If we examine the result obtained above, along with the results obtained by Alam and Roohi (2003) and Khan and Islam (2007a), we find that

(i). Chances of catastrophe increase with increase in λ for $m=\frac{\theta}{\beta}<1$ implying that so long as maximum strength is less than the average stress, exponential distribution of the stress is our best bet.

(ii). Chances of catastrophe decreases with increase in λ for $m=\frac{\theta}{\beta}>1$ implying that we must have maximum strength greater than the average stress if λ increases. In order to facilitate this we may find the suitable value of m so that the probability of a disaster may be pre-determined value of $\alpha=P(X>\theta)$ that may be regarded as tolerance level of the item because chances of a disaster do not vanish.

Naturally, we would like this α to be as small as possible. Clearly, (2.5) implies that for a fixed value of α, m_3 is given by

$$m_3 = \sqrt{\frac{-\ln\alpha}{0.712}} \tag{2.7}$$

Similarly, (2.6) implies that for fixed values of α, m_4 is given by

$$m_4 = \sqrt{\frac{-\ln\alpha}{0.675}} \tag{2.8}$$

Table 3.2 and Table 3.3 give the values of m_3 and m_4 respectively, for selected tolerance levels α.

Table 3.2: For $\lambda = 3$, values of m for selected tolerance levels α.

α	0.1	0.05	0.02	0.01	0.001	0.0001	0.00001
m_3	1.4788	1.6143	1.7645	1.8631	2.1328	2.3474	2.5287

Table 3.3: For $\lambda = 4$, values of m for selected tolerance levels α.

α	0.1	0.05	0.02	0.01	0.001	0.0001	0.00001
m_4	1.3590	1.4514	1.5515	1.6161	1.7885	1.9219	2.0322

2.2 Facing strength reliabilities with respect to stresses

Our ultimate aim here is to find the reliability of the item under stress and strength having the statistical formulation expressed through (2.1) and (2.2).

Using the result of Nandi and Aich (1994), the strength reliability of an item is given by

$$R = P[Y > X] = \int_0^\infty x f(x) \{ \int_1^\infty g(vx) dv \} dx$$

In this particular case

$$R = \int_0^{m\beta} x \frac{\lambda x^{\lambda-1}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}}{\beta^{\lambda}} e^{-(\frac{x}{\beta})^{\lambda}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}} \left\{ \int_1^{m\beta/x} \left(\frac{p}{\theta}\right)\left(\frac{vx}{\theta}\right)^{p-1} dv \right\} dx$$

where $k = \frac{\theta}{\beta}$. It can be easily seen that

$$R = \int_0^{m\beta} x \frac{\lambda x^{\lambda-1}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}}{\beta^{\lambda}} e^{-(\frac{x}{\beta})^{\lambda}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}} \frac{px^{p-1}}{(m\beta)^p} \left[\int_1^{m\beta/x} (v)^{p-1} dv \right] dx$$

$$= \int_0^{m\beta} x \frac{\lambda x^{\lambda-1}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}}{\beta^{\lambda}} e^{-(\frac{x}{\beta})^{\lambda}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}} \frac{px^{p-1}}{(m\beta)^p} \left[\left(\frac{m\beta}{x}\right)^p - 1 \right] dx$$

$$= \int_0^{m\beta} \frac{\lambda x^{\lambda-1}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}}{\beta^{\lambda}} e^{-(\frac{x}{\beta})^{\lambda}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}} dx$$

$$- \int_0^{m\beta} \frac{1}{(m\beta)^p} \frac{\lambda x^{\lambda+p-1}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}}{\beta^{\lambda}} e^{-(\frac{x}{\beta})^{\lambda}[\Gamma(\frac{1}{\lambda}+1)]^{\lambda}} dx$$

$$= \left[1 - e^{-[m.\Gamma(\frac{1}{\lambda}+1)]^{\lambda}} \right] - \frac{1}{[m.\Gamma(\frac{1}{\lambda}+1)]^p} \int_0^{[m.\Gamma(\frac{1}{\lambda}+1)]^{\lambda}} t^{p/\lambda} e^{-t} dt,$$

$$\text{where } t = \left(\frac{x}{\beta}\right)^{\lambda} [\Gamma(\frac{1}{\lambda}+1)]^{\lambda} \qquad (2.9)$$

Now, if we let $\lambda = 1$, we get

$$R_1 = (1 - e^{-m}) - \frac{1}{m^p} \int_0^m t^p e^{-t} dt, \text{ where } t = \frac{x}{\beta} \qquad (2.10)$$

as obtained by Alam and Roohi, (2003) for exponential stress and strength having power function distribution.

Similarly, for $\lambda = 2$, we get

$$R_2 = [1 - e^{-\frac{\pi}{4}m^2}] - \frac{1}{\left(\frac{\pi}{4}m^2\right)^{p/2}} \int_0^{\frac{\pi}{4}m^2} t^{p/2} e^{-t}\, dt \,,$$

$$\text{where } t = \frac{\pi x^2}{4\beta^2} \qquad (2.11)$$

as obtained by Khan and Islam, (2007a) for Rayleigh stress and strength having power function distribution.

For shape parameter $\lambda = 3$, we get

$$R_3 = [1 - e^{-0.712 m^2}] - \frac{1}{[0.712 m^2]^{p/3}} \int_0^{0.712 m^2} t^{p/3} e^{-t}\, dt,$$

$$\text{where } t = 0.712 \frac{x^2}{\beta^2} \qquad (2.12)$$

and for shape parameter $\lambda = 4$, we get

$$R_4 = [1 - e^{-0.675 m^2}] - \frac{1}{[0.675 m^2]^{p/4}} \int_0^{0.675 m^2} t^{p/4} e^{-t}\, dt,$$

$$\text{where } t = 0.675 \frac{x^2}{\beta^2} \qquad (2.13)$$

It is not easy to evaluate R_3 and R_4 even for given values of p, m_3 and m_4. However, it can be approximated by using Table of incomplete Gamma function, provided by Karl Pearson (1957) or some appropriate Statistical Package. We obtain R_3 and R_4 for selected values of p, m_3 and m_4 as shown in Table 3.4 and Table 3.5.

Table 3.4, Strength exceeds Stress for $\lambda = 3$.

$m_3 \rightarrow$ $p\downarrow$	**1.0**	**1.25**	**1.50**	**1.75**	**2.0**	**2.25**	**2.50**	**2.75**	**3.0**
1.5	0.1922	0.3311	0.4645	0.5522	0.6395	0.6976	0.7418	0.7762	0.8036
3.0	0.2870	0.4540	0.6347	0.7469	0.8252	0.8767	0.9101	0.9325	0.9480
4.5	0.2958	0.5662	0.7126	0.8318	0.9032	0.9425	0.9642	0.9767	0.9842
6.0	0.3780	0.6369	0.7740	0.8773	0.9403	0.9697	0.9840	0.9909	0.9946
7.5	0.4518	0.6313	0.8063	0.9055	0.9593	0.9824	0.9919	0.9960	0.9980
9.0	0.4535	0.6757	0.8138	0.9211	0.9704	0.9892	0.9956	0.9982	0.9992
10.5	0.4543	0.6774	0.8351	0.9367	0.9775	0.9925	0.9975	0.9991	0.9996
12.0	0.5053	0.6736	0.8542	0.9404	0.9816	0.9947	0.9984	0.9997	0.9999

Table 3.5, Strength exceeds Stress for $\lambda = 4$.

$m_4 \rightarrow$ $p\downarrow$	**1.0**	**1.25**	**1.50**	**1.75**	**2.0**	**2.25**
2.0	0.2257	0.3809	0.5411	0.6579	0.7379	0.7930
4.0	0.3267	0.5280	0.7177	0.8425	0.9074	0.9422
6.0	0.3434	0.6033	0.8091	0.9170	0.9625	0.9815
8.0	0.3448	0.6558	0.8577	0.9509	0.9828	0.9933
10.0	0.4250	0.6985	0.8836	0.9682	0.9913	0.9973
12.0	0.4253	0.7018	0.9008	0.9777	0.9952	0.9988
14.0	0.4458	0.7377	0.9134	0.9836	0.9972	0.9999

3. Discussion and example

We assume that it is possible to manufacture items having a probability distribution of its strength with parameters adjusted to a desired level. Alternatively, if the strength of an item is known to follow a particular probability distribution then the relevant components can be so designed that the parameters of the probability distribution are at a desired level. It is the performance of the components of an item that is reflected in the parameters of the probability distribution of the strength of the item. Hence it should be possible to redesign/readjust/reassemble the components so as to bring the parameters of the probability distribution of the strength of the item at a desired level. For example, the pick-up of the engine of a vehicle can be increased or decreased by designing methods. Manufacturing of an item with its strength having a power function as its failure model, may use an upper limit of θ, for example capacity of accelerating an engine must have subject to maximum possible speed. Further, not only the values of the parameter p and θ are either known or may be approximated through usual statistical techniques, we may also redesign the equipment with required values of these parameters (at least approximately). As regards the stress, it is supposed to be beyond our control, but its probability density function may be approximated through a proper choice of β. Without loss of generality, one may assume that the average stress is 1 (one), i.e., m is equal to the maximum strength. Now referring to the example given in Alam and Roohi (2003), we see that for exponential stress $(\lambda = 1)$, to design a product to achieve 99% reliability we have to choose the values of $p = 6$ and $m = 6$ with the minimum cost factor $6C_1 + 6C_2$. In Khan and Islam (2007a) for Rayleigh stress $(\lambda = 2)$, we get the same reliability for $p = 8$, $m = 3$ with the minimum cost factor $8C_1 + 3C_2$. For Weibull distribution when $\lambda = 3$, we achieve the same reliability for values of $p = 7.5$ and $m = 2.5$ with the minimum cost factor is $7.5C_1 + 2.5C_2$. And for $\lambda = 4$, we get the same reliability for $p = 10$ and $m = 2$ with the minimum cost factor $10C_1 + 2C_2$. The above results show that when the value of shape parameter (λ) increases, the probability of disaster decreases rapidly. So for an item when its strength follows a Power function distribution, it is better to use Weibull distribution as stress distribution. We thus conclude that Weibull probability distribution is most suitable to design a product, with the pre-determined probability and minimum cost too.

*SYSTEM RELIABILITY WITH SINGLE STRENGTH AND MULTI-COMPONENT STRESS MODEL

1. Introduction

Reliability of a consumer or engineering product has always been a big concern for manufacturers. Customer expects to use products over a certain period of time without any problems. This leads to the problem of addressing the risks involved in perceived quality and reliability levels and if possible, control or reduce them at the design phase. In the early phases of the design development, actual field information of the product function will not be available. Therefore, predictive models for the product design must be developed in order to allow an assessment of its behaviour in the field. It is only through this that it is possible to realize the early identification and resolution of potential quality and reliability problems.

In the process of developing a new product, the engineer is often faced with the task of designing a system that conforms to a set of reliability specifications. The engineer is given the goal for the system and must then develop a design that will achieve the desired reliability of the system, while performing all of the system's intended functions at a minimum cost. This involves a balancing act of determining how to distribute reliability to the components in the system so the system will meet its reliability goal while at the same time ensuring that the system meets all of the other associated performance specifications.

The concept of reliability is used in a variety of business and industrial

* Part of the results of this chapter is contained in Islam and Khan (2007a).

settings. In general, the concept of reliability is applied where it is important to achieve the same results again and again. A manufacturing process is said to be reliable when it achieves the same results, within defined limits, each time it occurs. An automobile, or other type of product, is reliable if it performs consistently and up to expectations. Reliability is measured by results. It is the yardstick against which performance is measured and evaluated. Reliability is applied to the performance of individuals, products, processes, and data, among other things. Reliable performance in all of these areas is critical to successful business planning and results. In order for a business to be successful, all of its components must be reliable. Since reliability does not necessarily mean perfection, constant attention is paid to improving the reliability of a wide range of manufacturing functions. The reliability of such processes directly affects the profitability of a manufacturing firm as well as the reliability of its products.

Product reliability is important not only to the manufacturer, but also to the consumer. When consumers purchase products, they have certain expectations as to how well those products will perform and for how long. When manufacturers offer product warranties, they are standing behind the reliability of their products. A computer that has a three-year warranty can be expected to be more reliable than one with only a two-year warranty. During the warranty period, the manufacturer generally assumes the cost of any repairs or defects and, in some cases, may even replace the product at no additional charge.

It has been observed that both, the capacities of structures or systems and the loads are probabilistic due to variations in material properties like production process, geometrical dimensions etc. In reliability engineering, the workload is interpreted as ‘Stress’ applied on a system and capacity as the 'Strength' of the system. Reliability evaluation in the frame work of probabilistic 'Stress-Strength' models is based on the following expression of reliability function for a given mission time i.e. $R = P(Y > X)$, where the random variable X ('Stress') is the peak value of stress and Y is its strength. As such, when the distribution function of both strength and stress variables are known, the studies like Kapoor and Lamberson (1977), Jaisingh (1988), Chaurch and Harris (1970), Mazumdar (1970), Dowton (1973), Beg and Singh (1979), Reiser and Guttman (1986), Nandi and Aich (1994), Pandey and Borhanuddin (1990) and Rehamn *et al.* (2000) have analytically determined the system reliability.

Following Alam and Roohi (2002, 2003) and Khan and Islam (2007a, b), in this chapter, we have considered the reliability of a system when $n-$Stresses with exponential probability distributions act on a single strength component. Thus the stress component has been decomposed in the form of a multi-component system. An electric engine pulling a number of compartments of identical nature may be a practical example representing the situation. However, in real life, in a residential locality a number of electric power connections (stresses) are attached with the single transformer (strength) of that locality, and if the load of the electric current exceeds the strength of the transformer, this will result in breakdown of the transformer. Let us take another example; the number of users of mobile cellular network has increased rapidly in recent years. The load of the network varies in different areas and times, exceeding network capacity occasionally. When the number of mobile users (stresses) exceeds the network capacity (strength), then the network congestion problem may occur. Here reliability is network's ability to perform a designated set of functions under certain conditions for specified operational times.

2. System reliability with single strength

Let, the strength variable Y follows the Power function distribution with probability density function

$$g(y)=\left(\frac{c}{\theta}\right)\left(\frac{y}{\theta}\right)^{c-1}, \quad 0<y<\theta,\ c>0 \tag{2.1}$$

and stress variables $(X_i\,; i=1, 2, 3,\dots,n)$ are assumed to be independent and identically distributed (*iid*) as exponential with probability density function

$$f(x_i)=\frac{1}{\beta}\ e^{-x_i/\beta}, \quad x_i,\beta>0 \tag{2.2}$$

Now, if Y is the strength of a system and $X_1, X_2,\dots, X_n$ are *iid* stresses acting on a single strength component of the system, then the reliability of the system, R_n can be defined as

$$R_n=P(Y>X_1+X_2+\cdots+X_n) \tag{2.3}$$

Let us suppose that two random stresses X_1 and X_2 are acting on a single strength component Y. Then the density function $f(x)$ of total stress, $X=X_1+X_2$ can be obtained by using the convolution formula

$$f(x)=\int_0^x f(x_1)f(x-x_1)\,dx_1\text{, where } X_2=X-X_1 \tag{2.4}$$

Using (2.2), (2.4) becomes

$$f(x)=\int_0^x\left(\frac{1}{\beta}e^{-x_1/\beta}\right)\left(\frac{1}{\beta}e^{-(x-x_1)/\beta}\right)dx_1$$

$$f(x)=\frac{x}{\beta^2}e^{-x/\beta}$$

Now, by definition

$$R_2 = P(Y > X_1 + X_2)$$

Similarly, if we take three stresses, X_1, X_2 and X_3 impinging on a single strength Y, then letting

$X = X_1 + X_2 + X_3$, and $X^* = X_1 + X_2$ or $X = X^* + X_3$

we have

$$f(x^*) = \frac{x^*}{\beta^2} e^{-x^*/\beta}$$

$$\begin{aligned} f(x) &= \int_0^x f(x^*) f(x_3)\, dx^* \\ &= \int_0^x f(x^*) f(x - x^*)\, dx^* \\ &= \int_0^x \left(\frac{x^*}{\beta^2} e^{-x^*/\beta}\right)\left(\frac{1}{\beta} e^{-(x-x^*)/\beta}\right) dx^* \\ &= \frac{x^2}{2\beta^3} e^{-x/\beta} \end{aligned}$$

Therefore

$$R_3 = P(Y > X_1 + X_2 + X_3)$$

and finally, in general, if we take n random stresses $X_1, X_2, ..., X_n$, and let $Z = X_1 + X_2 + \cdots + X_n$, then the combined effect of the stresses is

$$f(z) = \frac{1}{\Gamma n}\left(\frac{n}{\beta}\right)^n z^{n-1} e^{-(n z/\beta)}, \quad \beta, z > 0 \tag{2.5}$$

which is the well-known gamma distribution with parameters β and n.

2.1 Disaster reliability of the system

Since maximum possible value of Y (strength) is θ and Y cannot exceeds Z, if Z exceeds θ. In order to minimize the chance of a disaster in the present setup we proceed as before to find $P[Z > \theta]$, the unreliability of the strength Y against the stress Z as follows:

$$P[Z > \theta] = \int_{\theta}^{\infty} \frac{1}{\Gamma n} \left(\frac{n}{\beta}\right)^{n} z^{n-1} e^{n z / \beta} dz$$

$$= \frac{1}{\Gamma n} \left(\frac{n}{\beta}\right)^{n} \int_{\theta}^{\infty} z^{n-1} e^{n z / \beta} dz$$

Let $\frac{n z}{\beta} = t$ then $\frac{n}{\beta} dz = dt$

$$= \frac{1}{\Gamma n} \int_{n\theta/\beta}^{\infty} t^{n-1} e^{-t} dt$$

$$= 1 - \frac{1}{\Gamma n} \int_{0}^{n\theta/\beta} t^{n-1} e^{-t} dt$$

For fixed θ and known β, let $m = \frac{\theta}{\beta}$, then

$$\alpha = P[Z > \theta] = 1 - \frac{1}{\Gamma n} \int_{0}^{n m} t^{n-1} e^{-t} dt \qquad (2.6)$$

$$\alpha = 1 - I(m\sqrt{n},\ n) \qquad (2.7)$$

where $I(m\sqrt{n},\ n)$ is the incomplete gamma function.

Noting that for $n = 1$ in (2.7), we get

$$\alpha = P[Z > \theta] = e^{-m}$$

which is same as the result obtained by Alam and Roohi, (2003) for exponential stress.

For different values of n, Table 3.1 shows the probability of disaster for the system.

Table 3.1: $P[Z > \theta]$, **the probability of disaster for system**

m	$\alpha = P[Z > \theta]$	$\alpha = P[Z > \theta]$	$\alpha = P[Z > \theta]$
	$n = 2$	$n = 4$	$n = 5$
0.1	0.9992	0.9998	0.9999
0.2	0.9840	0.9977	0.9983
0.25	0.9667	0.9941	0.9959
0.50	0.8767	0.9235	0.9437
0.75	0.7487	0.7526	0.7589
1.00	0.5632	0.5374	0.5480
1.25	0.3973	0.3436	0.3191
1.50	0.2619	0.2013	0.1629
1.75	0.1936	0.1100	0.07512
2.00	0.1378	0.0568	0.03701
2.25	0.08578	0.02809	0.01740
2.50	0.05925	0.0133	0.006687
2.75	0.03561	0.006164	0.002907
3.00	0.024067	0.002767	0.0010315
3.50	0.009378	0.0005224	0.00014180
4.00	0.003073	0.0000919	0.0000179
4.50	0.0011278	0.0000153	0.0000017
5.00	0.0004057	0.0000024	0.0000002
5.50	0.0001436	0.0000004	
6.00	0.0000501		

By choosing α suitably, the parameters of stress and strength distribution can be determined to ensure the desired level of reliability. Clearly (2.7) shows that for a fixed value of α, m is given by

$$I(m\sqrt{n},\ n) = 1 - \alpha \tag{2.9}$$

and approximate values of m can be obtained form the incomplete gamma table (Pearson, 1957).

Table: 3.2, 3.3 and 3.4 gives the values of m, for different values of n and selected tolerance levels α.

Table 3.2: For $n = 2$, values of m for selected tolerance levels α

α	0.1	0.05	0.02	0.01	0.001	0.0001	0.00001
m	2.1213	2.5455	3.0405	3.4648	4.5254	5.6568	6.7882

Table 3.3: For $n = 4$, values of m for selected tolerance levels α

α	0.1	0.05	0.02	0.01	0.001	0.0001	0.00001
m	1.7750	2.0250	2.3750	2.5750	3.3250	3.9750	4.6500

Table 3.4: For $n = 5$, values of m for selected tolerance levels α

α	0.1	0.05	0.02	0.01	0.001	0.0001	0.00001
m	1.6770	1.9006	2.2137	2.3925	3.0186	3.5533	4.1143

2.2 Strength reliabilities of the system with respect to n – stresses

Our ultimate aim here is to find the reliability of the system under stresses having the statistical formulation expressed through (2.1) and (2.5).

The strength reliability of the item is given by

$$R = P\,[Y > X\,] = \int_0^{\infty} x\,f\,(x)\,\{\int_1^{\infty} g\,(vx)\,dv\,\}\,dx$$

However, in this case

$$R_n = \int_0^{m\beta} z\,\frac{1}{\Gamma n}\left(\frac{n}{\beta}\right)^n z^{n-1} e^{-nz/\beta}\,\{\int_1^{m\beta/z}\left(\frac{c}{\theta}\right)\left(\frac{vz}{\theta}\right)^{c-1} dv\}\,dz$$

where $m = \dfrac{\theta}{\beta}$. It can be easily seen that

$$= \int_0^{m\beta} z\,\frac{1}{\Gamma n}\left(\frac{n}{\beta}\right)^n z^{n-1} e^{-nz/\beta}\,\{\int_1^{m\beta/z}\left(\frac{c}{m\beta}\right)\left(\frac{vz}{m\beta}\right)^{c-1} dv\}\,dz$$

$$= \int_0^{m\beta} z\,\frac{1}{\Gamma n}\left(\frac{n}{\beta}\right)^n z^{n-1} e^{-n\,z/\beta}\,\frac{z^{c-1}}{(m\beta)^c}\left[\left(\frac{m\beta}{z}\right)^c - 1\right] dz$$

$$= \frac{1}{\Gamma n}\left(\frac{n}{\beta}\right)^{n} \int_{0}^{m\beta} z^{n-1} e^{-nz/\beta} dz$$

$$- \frac{1}{\Gamma n}\left(\frac{n}{\beta}\right)^{n} \frac{1}{(m\beta)^{c}} \int_{0}^{m\beta} z^{n+c-1} e^{-nz/\beta} dz$$

Let $\frac{nz}{\beta} = t$ then $dz = \frac{\beta}{n} dt$

$$R_n = \frac{1}{\Gamma n} \int_{0}^{mn} t^{n-1} e^{-t} dz - \frac{1}{\Gamma n} \frac{1}{(mn)^{c}} \int_{0}^{mn} t^{n+c-1} e^{-t} dz \qquad (2.11)$$

The above expression (2.11) is too much complicated and comes in as incomplete gamma function. So it is not easy to evaluate R_n even for given values of c, n and m. However, it can be approximated by using Table of incomplete gamma function. We obtain R_n for selected values of c, n and m as shown in Tables 3.5, 3.6 and 3.7.

For $n = 1$ i.e. when single stress is working against a single strength, we get

$$R_1 = [1 - e^{-m}] - \frac{1}{(m)^{c}} \int_{0}^{m} t^{c-1} e^{-t} dz \qquad (2.12)$$

as obtained by Alam and Roohi (2003) for exponential stress and power function distribution as strength.

Similarly, for $n = 2$ i.e. when double stress is working against a single strength, we get

$$R_2 = \frac{1}{\Gamma 2} \int_{0}^{2m} t^{2-1} e^{-t} dz - \frac{1}{\Gamma 2} \frac{1}{(2m)^{c}} \int_{0}^{2m} t^{2+c-1} e^{-t} dz \qquad (2.13)$$

Table 3.5: Strength exceeds double stress

$m \rightarrow$ $c \downarrow$	**2.5**	**3.0**	**3.5**	**4.0**	**4.5**	**5.0**	**5.5**	**6.0**	**6.5**
0.5	0.3807	0.5900	0.6207	0.7235	0.6668	0.6837	0.6984	0.7113	0.7226
1.0	0.8175	0.8258	0.8538	0.8737	0.8887	0.8999	0.9090	0.9166	0.9230
1.5	0.8869	0.9202	0.9405	0.9536	0.9624	0.9681	0.9725	0.9759	0.9786
2.0	0.9192	0.9567	0.9724	0.9817	0.9868	0.9896	0.9916	0.9930	0.9940
2.5	0.9329	0.9708	0.9841	0.9914	0.9949	0.9964	0.9973	0.9979	0.9983
3.0	0.9371	0.9759	0.9883	0.9947	0.9976	0.9985	0.9990	0.9993	0.9995
3.5	0.9394	0.9778	0.9898	0.9957	0.9983	0.9991	0.9995	0.9997	0.9998
4.0	0.9402	0.9785	0.9903	0.9966	0.9987	0.9994	0.9997	0.9998	0.9999

and for $n = 4$ i.e. when four stress are acting against a single strength, we get

$$R_4 = \frac{1}{\Gamma 4} \int_0^{4m} t^{4-1} e^{-t} \, dz - \frac{1}{\Gamma 4} \frac{1}{(4m)^c} \int_0^{4m} t^{4+c-1} e^{-t} dz \qquad (2.14)$$

Table 3.6: Strength exceeds four stress

$m \rightarrow$ $c \downarrow$	**1.5**	**2.0**	**2.25**	**2.5**	**3.0**	**3.5**	**4.0**	**4.5**
0.5	0.5066	0.6241	0.6551	0.6780	0.7102	0.7325	0.7499	0.7612
1.0	0.6967	0.8361	0.8691	0.8909	0.9148	0.9282	0.9374	0.9402
1.5	0.7618	0.9068	0.9389	0.9570	0.9735	0.9804	0.9843	0.9882
2.0	0.7861	0.9315	0.9615	0.9776	0.9905	0.9944	0.9960	0.9974
2.5	0.7944	0.9393	0.9686	0.9838	0.9953	0.9981	0.9989	0.9993
3.0	0.7973	0.9419	0.9708	0.9858	0.9966	0.9991	0.9996	0.9997
3.5	0.7933	0.9427	0.9716	0.9863	0.9970	0.9993	0.9998	0.9998
4.0	0.7985	0.9430	0.9718	0.9865	0.9971	0.9994	0.9999	0.9999

For $n = 5$ i.e. when five stress are impinging against a single strength, we get

$$R_5 = \frac{1}{\Gamma 5}\int_0^{5m} t^{5-1}\, e^{-t}\, dz - \frac{1}{\Gamma 5}\frac{1}{(5m)^c}\int_0^{5m} t^{5+c-1}\, e^{-t} dz \qquad (2.15)$$

Table 3.7: Strength exceeds five stress

$m \rightarrow$ $c\downarrow$	**1.5**	**1.75**	**2.0**	**2.25**	**2.5**	**3.0**	**3.5**
0.5	0.5340	0.6178	0.6610	0.6922	0.7138	0.7413	0.7608
1.0	0.7257	0.8207	0.8669	0.8974	0.9148	0.9326	0.9427
1.5	0.7339	0.8841	0.9294	0.9577	0.9713	0.9819	0.9862
2.0	08067	0.9046	0.9492	0.9754	0.9872	0.9946	0.9965
2.5	0.8092	0.9112	0.9547	0.9805	0.9916	0.9979	0.9990
3.0	0.8109	0.9131	0.9564	0.9820	0.9928	0.9987	0.9996
3.5	0.8160	0.9137	0.9569	0.9824	0.9931	0.9989	0.9998
4.0	0.8168	0.9140	0.9574	0.9831	0.9935	0.9992	0.9999

3. An illustrative example and discussion

If a system with five components in series has a reliability objective of 90% for a given operating time, the uniform allocation of the objective to all components could require each component to have a reliability of 98% for the specified operating time, since $0.98^5 \cong 0.90$. While this manner of allocation is easy for calculations, it is generally not the best way to allocate reliability for a system facing a stress. The optimum method of allocating reliability in that case, would take into account the cost or relative difficulty of improving the reliability of different subsystems or components. Clearly, in such cases the table provided in section (2.2) may be useful, if available, otherwise the results mentioned in section

(2.2) may be utilized for particular cases. For example, the above example corresponds to $n=5$, Table 3.7, is therefore the relevant table. Even for the simple case of $c=1$ (uniform strength), we get $m=2.25$ approximately, which implies that the components should have 2.25 times average stress as that of the single stress so as to have a 90% reliability against it.

*RELIABILITY COMPUTATION FOR α-DISTRIBUTED STRENGTH AND STRESS USING GAUSSIAN FUNCTION

1. Introduction

In this chapter an alternative method is discussed to evaluate reliability of a system in stress-strength situations. Further, some of its variants are also discussed.

Kattan (1996) has considered the density function

$$f(x)=\frac{1}{\Phi(1/\sqrt{\alpha_1})}\frac{c_1}{\sqrt{2\pi\alpha_1}}\frac{1}{x^2}\exp\left[-\frac{1}{2\alpha_1}\left(1-\frac{c_1}{x}\right)^2\right],\ x>0, \alpha_1, c_1>0 \tag{1.1}$$

as the probability distribution function of strength (X) of an item and described it as half alpha distribution with parameters α_1 and c_1.

Similarly,

$$g(y)=\frac{1}{\Phi(1/\sqrt{\alpha_2})}\frac{c_2}{\sqrt{2\pi\alpha_2}}\frac{1}{y^2}\exp\left[-\frac{1}{2\alpha_2}\left(1-\frac{c_2}{y}\right)^2\right],\ y>0, \alpha_2, c_2>0 \tag{1.2}$$

has been defined as probability density function of stress (Y) that the item may suffer.

Assuming the strength (X) and stress (Y) to be independent random variables, the strength reliability of an item is given by

$$R = P[X>Y] = \int_0^\infty\int_y^\infty f(x)\,g(y)\,dx\,dy \tag{1.3}$$

* Part of the results of this chapter is contained in Khan and Islam (2007c).

Kattan (1996) has used the usual mathematical tools to derive the expression for R for given $f(\cdot)$ and $g(\cdot)$. However, Nandi and Aich (1994) have used the relation

$$R = P[X > Y] = \int_0^\infty y\, g(y) \left\{ \int_1^\infty f(vy)\, dv \right\} dy" \tag{1.4}$$

to simplify the expression.

We have utilized (1.4) to calculate $P[X > Y]$ for the distributions considered in (1.1) and (1.2).

2. Reliability Computation

We have

$$R = P[X > Y] = \frac{c_1 c_2}{2\pi\sqrt{\alpha_1 \alpha_2}} \frac{1}{\Phi(1/\sqrt{\alpha_1})\Phi(1/\sqrt{\alpha_2})}$$

$$\times \int_0^\infty y \frac{1}{y^2} \exp\left[-\frac{1}{2\alpha_2}\left(1 - \frac{c_2}{y}\right)^2 \right] \left\{ \int_1^\infty \frac{1}{v^2 y^2} \exp\left[-\frac{1}{2\alpha_1}\left(1 - \frac{c_1}{vy}\right)^2 \right] dv \right\} dy$$

$$= \frac{c_1 c_2}{2\pi\sqrt{\alpha_1 \alpha_2}} \frac{1}{\Phi(1/\sqrt{\alpha_1})\Phi(1/\sqrt{\alpha_2})} \int_0^\infty \frac{1}{y^2} \exp\left[-\frac{1}{2\alpha_2}\left(1 - \frac{c_2}{y}\right)^2 \right]$$

$$\times \left\{ \int_1^\infty \frac{1}{v^2 y} \exp\left[-\frac{1}{2\alpha_1}\left(1 - \frac{c_1}{vy}\right)^2 \right] dv \right\} dy$$

It can be easily seen that

$$\int_1^\infty \frac{1}{v^2 y} \exp\left[-\frac{1}{2\alpha_1}\left(1 - \frac{c_1}{vy}\right)^2 \right] dv = \int_{\frac{1}{\sqrt{\alpha_1}}\left(1 - \frac{c_1}{y}\right)}^{1/\sqrt{\alpha_1}} \frac{\sqrt{\alpha_1}}{c_1} e^{-u^2/2}\, du$$

where $u = \frac{1}{\sqrt{\alpha_1}}\left(1-\frac{c_1}{vy}\right)$ then $\frac{\sqrt{\alpha_1}}{c_1}du = \frac{1}{v^2 y}dv$

$$\int_1^\infty \frac{1}{v^2 y}\exp\left[-\frac{1}{2\alpha_1}\left(1-\frac{c_1}{vy}\right)^2\right] dv$$

$$= \frac{\sqrt{2\pi\alpha_1}}{c_1}\left[\Phi\left(1/\sqrt{\alpha_1}\right) - \Phi\left\{\frac{1}{\sqrt{\alpha_1}}\left(1-\frac{c_1}{y}\right)\right\}\right]$$

$$= \frac{c_1 c_2}{2\pi\sqrt{\alpha_1\alpha_2}}\frac{1}{\Phi(1/\sqrt{\alpha_1})\Phi(1/\sqrt{\alpha_2})}\int_0^\infty \frac{1}{y^2}\exp\left[-\frac{1}{2\alpha_2}\left(1-\frac{c_2}{y}\right)^2\right]$$

$$\times\frac{\sqrt{2\pi\alpha_1}}{c_1}\left[\Phi\left(1/\sqrt{\alpha_1}\right) - \Phi\left\{\frac{1}{\sqrt{\alpha_1}}\left(1-\frac{c_1}{y}\right)\right\}\right]dy$$

$$= \frac{c_2}{\sqrt{2\pi\alpha_2}}\frac{1}{\Phi(1/\sqrt{\alpha_1})\Phi(1/\sqrt{\alpha_2})}\int_0^\infty \frac{1}{y^2}\exp\left[-\frac{1}{2\alpha_2}\left(1-\frac{c_2}{y}\right)^2\right]$$

$$\times\left[\Phi\left(1/\sqrt{\alpha_1}\right) - \Phi\left\{\frac{1}{\sqrt{\alpha_1}}\left(1-\frac{c_1}{y}\right)\right\}\right]dy$$

and finally, we get

$$R = \frac{1}{\sqrt{2\pi}\,\Phi(1/\sqrt{\alpha_1})\Phi(1/\sqrt{\alpha_2})}$$

$$\times\int_{-\infty}^{1/\sqrt{\alpha_2}} e^{-t^2/2}\,dt\left[\Phi\left(1/\sqrt{\alpha_1}\right) - \Phi\left\{\frac{1}{\sqrt{\alpha_1}}\left(1-\frac{c_1}{c_2}(1-\sqrt{\alpha_2}\,t)\right)\right\}\right]$$

$$=\frac{\Phi(1/\sqrt{\alpha_1})}{\Phi(1/\sqrt{\alpha_1})\Phi(1/\sqrt{\alpha_2})}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1/\sqrt{\alpha_2}} e^{-t^2/2}\,dt$$

$$-\frac{1}{\Phi(1/\sqrt{\alpha_1})\Phi(1/\sqrt{\alpha_2})}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1/\sqrt{\alpha_2}} e^{-t^2/2}\,dt$$

$$\times\left[\Phi\left\{\frac{1}{\sqrt{\alpha_1}}\left(1-\frac{c_1}{c_2}(1-\sqrt{\alpha_2}\,t\right)\right\}\right]$$

$$=\frac{1}{\Phi(1/\sqrt{\alpha_2})}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{1/\sqrt{\alpha_2}} e^{-t^2/2}\,dt$$

$$-\frac{1}{\Phi(1/\sqrt{\alpha_1})\Phi(1/\sqrt{\alpha_2})}\int_{-\infty}^{1/\sqrt{\alpha_2}}\phi(t)\,dt$$

$$\times\left[\Phi\left\{\frac{1}{\sqrt{\alpha_1}}\left(1-\frac{c_1}{c_2}(1-\sqrt{\alpha_2}\,t\right)\right\}\right]$$

$$=1-\frac{1}{\Phi(1/\sqrt{\alpha_1})\Phi(1/\sqrt{\alpha_2})}\int_{-\infty}^{1/\sqrt{\alpha_2}}\Phi(a+bt)\,d\Phi(t)$$

where $a=\frac{1}{\sqrt{\alpha_1}}\left(1-\frac{c_1}{c_2}\right)$, $b=\frac{c_1}{c_2}\sqrt{\frac{\alpha_2}{\alpha_1}}$.

where $\Phi(t)$ and $\Phi(a+bt)$ are non-decreasing and left continuous functions of bounded variation.

Kattan has shown that

(i). If $c_1=c_2=c$ and $\alpha_1=\alpha_2=\alpha$, then $R=1/2$

This is obvious as in a competition between two matching forces; probability of one exceeding the other is always half. This also implies that $P(X<Y)$ is also half.

(ii). If $c_1=c_2=c$ and $\alpha_1\neq\alpha_2$, then $a=0$ and R is independent of c.

It clearly shows that in defining the life of a component, mean rate of wear and its initial value doesn't matter in this particular case.

We further consider the following limiting cases:

(i). If either $c_1 \to 0$ or $c_2 \to \infty$ then $a = \frac{1}{\sqrt{\alpha_1}}$, $b = 0$ and

$$R = 1 - \frac{1}{\Phi(1/\sqrt{\alpha_1})\Phi(1/\sqrt{\alpha_2})} \int_{-\infty}^{1/\sqrt{\alpha_2}} \Phi\left(1/\sqrt{\alpha_1}\right) d\Phi(t)$$

$$= 1 - \frac{\Phi\left(1/\sqrt{\alpha_1}\right)}{\Phi(1/\sqrt{\alpha_1})\Phi(1/\sqrt{\alpha_2})} \int_{-\infty}^{1/\sqrt{\alpha_2}} \phi(t)\,dt$$

$$= 1 - 1$$

$$= 0$$

(ii). If $\alpha_1 \to \infty$ then $a \to 0$ & $b \to 0$ and

$$R = 1 - \frac{1}{\Phi(0)\Phi(1/\sqrt{\alpha_2})} \int_{-\infty}^{1/\sqrt{\alpha_2}} \Phi(0)\, \phi(t)\,dt$$

$$= 1 - 1$$

$$= 0$$

and if $\alpha_1 \to 0$ then $a \to \infty$ & $b \to \infty$ and again $R = 0$.

*BAYESIAN ANALYSIS OF SYSTEM AVAILABILITY WITH GEOMETRIC FAILURE LAW IN LIFE TESTING

1. Introduction

An item functioning until its failure is 'available' before its failure. If the item is amenable to maintenance i.e. it will function once again after due maintenance, the item is 'available' except for the period during which it is under repair.

A system configuration is a logical combination of its components. Some basic system configurations are in general, k-out of-m having subsystems in series and/or parallel. The components arranged in these models are assumed to be identical and they function independently. Let $X_1, X_2, ..., X_m$ denote the lifetime of m components of a system. Let X_i's be independently, identically distributed (*iid*) random variables. A typical k-out of-m system consists of m components and operates as long as any of its $k (\le m)$ components operate, and that may be regarded as the 'availability' of the system. Obviously, for $k = m$, the system reduces to a series system, which fails as soon as any of its components fails, and for $k = 1$ the system reduces to parallel system which functions as long as any of its components operate.

Availability is an important measure in describing the performance of systems. It indicates the availability of system for use, taking into account failure and repair that apply to the system; hence it concerns all scientist and engineers engaged in developing systems. System availability has widely been studied in the literature because of its prevalence in industry.

* Part of the results of this chapter is contained in Islam and Khan (2007b).

Maintaining high or required level of availability is often an essential requisite. It is very important to reduce the number of failures to avoid unexpected sudden stops and to improve systems availability.

Usually the systems are analyzed with respect to their reliability characteristics assuming a continuous lifetime distribution. However, under certain cases when it is physically impossible or inconvenient to inspect a product's life length continuously until it fails, inspections are performed at specific intervals instead. In this case, time is not continuous and is measured on discrete scale and considers the number of successful cycles or operations before failure. For example: the bulb in Xerox machine lights up every time copy is taken, a spring may break after completing a certain number of cycles of 'to-and-fro' movements and also in packet transmission, when a packet is sent, the transmitter waits for an acknowledgement from receiver. If no acknowledgement is received the packet is retransmitted. The process continues until it is received successfully and similar situation occurs in sending data over an Ethernet. Collisions occur as other users are also trying to send information. Hence, the process is repeated until the first transmission without a collision. In such cases we define the random variable as the number of retransmission until success. Once a time period has been fixed as a unit for lifetime, the same unit has to be retained for the repair time, not only for the mathematical simplicity but also for the practical considerations. In order to find a suitable lifetime and repair time distribution, we have to look for the basic characteristics of available discrete distributions. It is easy to see that Geometric distribution has an important role in such analysis because it is the discrete analogue of exponential distribution and also because of its mathematical tractability.

For more insight in this regard, we refer to Yaqub and Khan (1981) who obtained parametric and non-parametric estimates of the reliability characteristics of this distribution. Bayesian estimation of mean life and reliability function for this life model has been studied by Bhattacharya and Kumar (1985) and Bhattacharya and Tyagi (1990). Maiti (1995) obtained the MLE, UMVUE and Bayes estimates of $P[X \leq Y]$ in geometric case.

It makes a great deal of practical sense to use all the information available, old and/or new, objective or subjective, when making decisions under uncertainty. This is especially true when the consequences of the decisions can have a significant impact, financial or otherwise. Most of us make everyday personal decisions this way, using an intuitive process based on our experience and subjective judgements. The classical statistical approach considers these parameters as fixed but unknown constant to be estimated using sample data taken randomly from the population of interest. Bayesian approach treats these population models as random quantities and makes good use of old information, or even subjective judgements, to construct a prior distribution model for these parameters, and then make use of current data to revise this starting assessment in the form of a posterior distribution model for the population model parameters. While the primary motivation to use Bayesian reliability methods is typically a desire to save on test time and materials cost, there are other factors that should also be taken into account. Recently Rehman *et al.* (2003) deal with the Bayesian availability analysis of an k-out-of-m system of operational data assuming exponential distribution for the lifetime of each component as also its repair time, although with different parameters. In this chapter, we obtain posterior analysis of system availability; Bayes point estimator and

Bayesian analysis of system availability with geometric failure distribution as well as repair time distribution and Beta prior distribution for the parameters of the lifetime distribution. Choice of Beta distribution for Bayesian analysis is imperative as it is a natural conjugate of geometric distribution.

2. Basic concepts and assumptions

a). Let the parameter q represent the probability of failure on a single trial and the random variable X represents the number of trials performed until failure. Failure time distribution for each component is a geometric distribution with probability mass function

$$P(x, q) = q\,(1-q)^{x-1}\,, \qquad 0 < q < 1,\ x = 1,2,3,........ \tag{2.1}$$

For such a component, mean time between failure (MTBF) is $1/q$.

b). Let r represent the probability of repair of each component within a single time period and let Y represent the number of time periods before the repair is required, then the probability mass function of Y is given by

$$P(y, r) = r\,(1-r)^{y-1}\,, \qquad 0 < r < 1,\ y = 1,2,3,........ \tag{2.2}$$

Here, mean time to repair (MTTR) is $1/r$.

c). The steady state component availability, denoted by A_c is the probability that the component is available in the long run, is given by:

$$A_c = \frac{MTBF}{MTBF + MTTR}$$

$$A_c = \frac{(1/q)}{(1/q + 1/r)} = \left(\frac{r}{r+q}\right) \tag{2.3}$$

d). In view of (2.3) and using simple probabilistic reasoning, the availability of (k, m) system, denoted by A_s, is given by

$$A_s = \sum_{i=k}^{m} \binom{m}{i} (A_c)^i\,(1-A_c)^{m-i}. \tag{2.4}$$

Here, failure and repair time distributions are as given in (2.1) and (2.2) and A_c is as given in (2.3). In particular

(i). For $k=1$ in (2.4), we get the parallel system availability as

$$A_{s1} = 1-(1-A_c)^m$$

$$= 1-\left(1-\frac{r}{r+q}\right)^m \tag{2.5}$$

(ii). For $k=m$ in (2.4), we get the availability for a series system as

$$A_{s2} = (A_c)^m = \left(\frac{r}{q+r}\right)^m \tag{2.6}$$

e). For Bayesian availability analysis, the prior distribution of the failure parameter q is assumed to be Beta with probability mass function given by:

$$g_1(q) = \frac{1}{B(a,b)} \; q^{a-1}\,(1-q)^{b-1}, \quad a,b>0 \tag{2.7}$$

Similarly, the prior distribution of the repair parameter r is another Beta distribution with probability mass function

$$g_2(r) = \frac{1}{B(c,d)} \; r^{c-1}\,(1-r)^{d-1}, \quad c,d>0 \tag{2.8}$$

f). During the course of operation, suppose U and V denote the number of failures and repairs respectively recorded between (0, n) trials. It may reasonably be assumed that the random variables U and V are independent.

3. Posterior analysis of system availability

Since the failure rate distribution for a component is geometric with MTBF $\frac{1}{q}$, for the total n trials, the probability of observing u failures will be given by Negative binomial distribution with probability mass function given by

$$P(u|\,q)=\binom{n-1}{u-1}\,q^{u}\,(1-q)^{n-u}\quad,\ \text{for } u=0,1,2,\,........ \qquad (3.1)$$

Similarly, the number of component repairs, v, performed during n trials will also follow Negative binomial distribution with probability mass function given by

$$P(v|\,r)=\binom{n-1}{v-1}\,q^{v}\,(1-q)^{n-v}\quad,\ \text{for } v=0,1,2,\,........ \qquad (3.2)$$

Now, the posterior distribution of q with respect to prior (2.7), given that u failures have been observed out of n trials, can be obtained as follows

$$\Pi_1(q\mid u)=\frac{P(u|\,q)\;g_1(q)}{\int_0^1 P(u|\,q)\;g_1(q)\;dq}$$

$$=\frac{\binom{n-1}{u-1}\;q^{u}\;(1-q)^{n-u}\;\frac{1}{B\,(a,\,b)}\;q^{a-1}\;(1-q)^{b-1}}{\frac{1}{B(a,\,b)}\binom{n-1}{u-1}\int_0^1 q^{u}\,(1-q)^{n-u}\;q^{a-1}\,(1-q)^{b-1}\;dq}$$

$$=\frac{q^{u+a-1}\;(1-q)^{n-u+b-1}}{\int_0^1 q^{u+a-1}\,(1-q)^{n-u+b-1}\;dq}$$

$$=\frac{1}{B\,(u+a,\,n-u+b)}\;q^{u+a-1}\;(1-q)^{n-u+b-1} \qquad (3.3)$$

which is a Beta density function with parameters $(u+a)$ and $(n-u+b)$. Similarly, the posterior distribution of r with respect to prior (2.8) is given by

$$\Pi_2(r \mid v) = \frac{P(v|r)\, g_2(r)}{\int_0^1 P(v|r)\, g_2(r)\, dr}$$

$$= \frac{1}{B\,(v+c,\, n-v+d)}\, r^{v+c-1}\, (1-r)^{n-v+d-1} \qquad (3.4)$$

which is also a Beta density function with parameters $(v+c)$ and $(n-v+d)$.

Now using the respective posterior distributions of q and r as obtained in (3.3) and (3.4) and also using the relationship $A_c = r/(r+q)$, the posterior distribution of A_c given u and v, is given by:

$$f(A_c|u,v) = \sum_{j=o}^{n-u+b-1} \binom{n-u+b-1}{j} (-1)^j \times \frac{B(n-v+d,\ u+v+a+c+j)}{[B(u+a,n-u+b),\, B(v+c,n-v+d)]} \times \frac{(1-A_c)^{u+a+j-1}}{A_c^{u+a+j+1}};\quad 0 < A_c < 1 \qquad (3.5)$$

4. Bayesian point estimators

By definition, the Bayes point estimators for q and r, say q^* and r^* respectively, are defined as the posterior expectations of their respective distributions, i.e.

$$q^* = E(q|a) = \int_0^1 q\, \Pi_1(q|u)\, dq$$

$$= \int_0^1 q\, \frac{1}{B(u+a,\, n-u+b)}\, q^{u+a-1} (1-q)^{n-u+b-1}\, dq$$

$$=\frac{1}{B(u+a,\,n-u+b)}\int_0^1 q^{u+a}\,(1-q)^{n-u+b-1}\,dq$$

$$=\frac{B(u+a+1,\,n-u+b)}{B(u+a,\,n-u+b)}$$

$$=\frac{\Gamma(u+a-1)\;\Gamma(n-u+b)\;\Gamma(n+a+b)}{\Gamma(n+a+b+1)\,\Gamma(u+a)\,\Gamma(n-u+b)}$$

$$q^*=E(q|a)=\frac{u+a}{n+a+b}, \tag{4.1}$$

and similarly,

$$r^*=E(r|\,v)=\int_0^1 r\,\Pi_1(r|\,u)\,dr$$

$$=\int_0^1 r\,\frac{1}{B(v+c,\,n-v+d)}\,r^{v+c-1}(1-r)^{n-v+d-1}\,dr$$

$$r^*=E(r|\,a\,)=\frac{v+c}{n+c+d} \tag{4.2}$$

Therefore, using the relation (2.3), the Bayes point estimator for availability A_c will be

$$A_c^*=\frac{r^*}{q^*+r^*}=\frac{(v+c)(n+a+b)}{(u+a)(n+c+d)+(v+c)(n+a+b)} \tag{4.3}$$

However, the Bayes point estimator of availability A_c can also be obtained by using the posterior distribution of A_c in (3.5).

5. Bayesian analysis of k – out of – m system

Finally, using the posterior distribution of A_c in (3.5) and assuming the squared error loss function, the Bayes point estimator of A_s, the availability of a $(k,\,m)$ system in (2.4), say A_s^*, can be obtained as

$$A_s^* = E[A_s \mid u, v] =$$

$$\int_0^1 \sum_{i=k}^{m} \binom{m}{i} A_c^i \, (1-A_c)^{m-i} \sum_{j=o}^{n-u+b-1} \binom{n-u+b-1}{j} (-1)^j$$

$$\times \frac{B(n-v+d,\ u+v+a+c+j)}{[B(u+a, n-u+b), B(v+c, n-v+d)]} \frac{(1-A_c)^{u+a+j-1}}{A_c^{u+a+j+1}} dA_c$$

$$= \sum_{j=o}^{n-u+b-1} \binom{n-u+b-1}{j} (-1)^j \frac{B(n-v+d,\ u+v+a+c+j)}{[B(u+a, n-u+b), B(v+c, n-v+d)]}$$

$$\times \sum_{i=k}^{m} \binom{m}{i} \ B(i-u-a-j,\ \ m+u+a+j-i) \tag{5.1}$$

For $k=1$ in (5.1), we get the availability of parallel system, say A_{s1}^* (for $b=1$),

$$A_{s1}^* = \sum_{j=o}^{n-u} \binom{n-u}{j} (-1)^j$$

$$\times \frac{B(n-v+d,\ u+v+a+c+j) B(m-u-a-j, u+a+j)}{[B(u+a, n-u)\, B(v+c, n-v+d)]} \tag{5.2}$$

$$\times \sum_{i=1}^{m} B(i-u-a-j, m+u+a+j-i)$$

This shows that for the successful working of geometric system, the parameter i is always greater than the sum of the parameters u, a, and j. Straightforward computation shows that if geometric failure unit are connected in parallel, then the reliability of the system does not obey geometric law in this particular case.

Similarly, for $k=m$ in (5.1), we get the availability of a series system, say A_{s2}^* (for $b=1$),

$$A_{s2}^{*} = \sum_{j=o}^{n-u} \binom{n-u}{j} (-1)^j \times \frac{B(n-v+d,\ u+v+a+c+j) B(m-u-a-j, u+a+j)}{[B(u+a, n-u)\ B(v+c, n-v+d)]} \tag{5.3}$$

Here it is important to note that Bayes estimate of availability of geometric distribution depends upon the number of trials while in exponential case (Rehman *et al.* 2003) is independent of time. In this case it is also imperative that number of components in the system is always greater than the sum of the parameters u, a, j i.e. $m > (u + a + j)$ so as to make the expression meaningful. A general discussion of the results is, therefore, not possible. In order to make the utility of the results clear, we consider a numerical illustration in the next section.

4. A numerical illustration

Bayes estimate of availability of a series system using a squared error loss function is given in (5.3). For analyzing the results, the posterior estimates of availability may be analyzed by keeping some of the parameters fixed and varying others. As an illustration, taking ($a=1$, $c=1$, $d=1$, $v=2$, $n=5$, $m=8$) and ($a=2$, $c=2$, $d=2$, $v=2$, $n=5$, $m=8$), and varying u, the variation in availability is shown in Table 6.1. This shows that posterior availability decreases uniformly as u, the number of failures recorded increases. Similarly, for ($a=1$, $c=1$, $d=1$, $u=1$, $n=5$, $m=8$) and ($a=2$, $c=2$, $d=2$, $v=2$, $n=5$, $m=8$) and varying v, the trend in availability is shown in Table 6.2.

Table 6.1: Bayes estimate of availability for variation in u.

u	Availability for $a=1$, $c=1$, $d=1$, $v=2$, $n=5$, $m=8$	Availability for $a=2$, $c=2$, $d=2$, $v=2$, $n=5$, $m=8$
1	0.074211	0.001628
2	0.041693	0.001131
3	0.021954	0.000662
4	0.009571	0.000204

Table 6.2: Bayes estimate of availability for variation in v.

v	Availability for $a=1$, $c=1$, $d=1$, $v=2$, $n=5$, $m=8$	Availability for $a=2$, $c=2$, $d=2$, $v=2$, $n=5$, $m=8$
1	0.043293	0.000267
2	0.078554	0.000353
3	0.123709	0.001049
4	0.182878	0.001638

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